MESHLESS METHODS: ALTERNATIVES FOR SOLVING 2D ELASTICITY PROBLEMS

Effendy TANOJO[1], Pamuda PUDJISURYADI[1]

ABSTRACT: In recent years meshless methods have been developed as alternatives to the well-known and widely used finite element method. The main objective in developing meshless methods is to overcome the difficulty of meshing and remeshing procedure of complex structural elements, which is one of the drawbacks of the finite element method. In meshless methods, element definition is no longer needed to discretize the problem domain, only nodal points definition and boundary conditions of the domain are necessary. This paper presents the concepts behind some prominent meshless methods: element-free Galerkin, meshless local Petrov-Galerkin, and finite point method. The derivations of these methods for solving 2D elasticity problems will also be presented and applied to a case problem. The results will then be compared with the ones obtained by finite element method and exact solutions.

KEYWORDS: Finite Element Method, meshless, Element-Free Galerkin, Meshless Local Petrov-Galerkin, Finite Point.

1. INTRODUCTION

Finite element method has been widely-known and well-accepted to be used in many engineering applications. However complex engineering problems, usually the ones involving a continuous change in the geometry of the problem domain caused by crack propagation or large deformation, cannot be solved easily using finite element method, as they require the a lot of "remeshing" procedure to ensure that the problem discretization still coincides with the geometry of the real structure as it continuously changes. Meshless methods provide alternatives to overcome this problem as they, unlike finite element method, do not require the definition of elements to discretize the problem domain.

To discretize the problem domain, meshless methods require only the definition of nodal points. The rediscretization procedure of the problem domain as it continuously changes is done by adding and/or moving nodal points only, without the need of redefining elements in each rediscretization. Thus the complex "meshing" and "remeshing" procedure is no longer needed, or in some cases significantly reduced.

During recent years, many meshless methods have been proposed and developed. Among them, the element-free galerkin [1], meshless local petrov-galerkin [2], and finite point method [3] can be considered as the prominent ones. This paper will present the concept and derivation of the three aforementioned methods for solving 2D elasticity problems. The results obtained from these methods, as they are applied to a test problem, will be compared to the ones obtained by using finite element method.

[1]*Lecturer, Dept. of Civil Engineering, Petra Christian University, Indonesia*

2. THEORY OF ELASTICITY

The governing differential equation required to be satisfied at all points in the domain, Ω, for two dimensional problem, homogeneous and isotropic material, is defined as

$$\sigma_{ij,j} + F_i = 0, \tag{1}$$

where i and j at the index represent the numbers 1 and 2 (or direction x and y in the two dimensional domain), $\sigma_{ij,j}$ are the stress components corresponding with the displacement field u_i, and F_i are the body force components acting the direction i at Ω. The boundary conditions of the domain, $\Gamma = \Gamma_u \cup \Gamma_t$, are as follows:

$$u_i = \overline{u}_i \quad \text{at } \Gamma_u, \tag{2a}$$

$$t_i = \overline{t}_i \quad \text{at } \Gamma_t, \tag{2b}$$

where the surface tractions are defined by $t_i = \sigma_{ij} \cdot n_j$, where n_j are the units normal to the boundary Γ, $\overline{u}_i$ and $\overline{t}_i$ are the prescribed values of the displacements and tractions at the boundary Γ_u and Γ_t, respectively. Γ_u symbolizes the essential boundary, Γ_t symbolizes the non-essential boundary.

3. MESHLESS METHODS

Generally meshless methods, to keep its local character, use a local approximation to represent the values of the unknown variables at some random nodal points with the trial function. Next, the most widely used local approximation for meshless methods, the moving least-squares approximation, will be discussed. It will be followed then by the concepts and derivations of the three aforementioned meshless methods for solving 2D elasticity problems.

3.1. MOVING LEAST-SQUARES (MLS) APPROXIMATION

In meshless methods, spatial discretization can be obtained by using MLS approximation which does not require the definition of elements, but uses nodal "selection" procedure instead. This approximation is introduced by Lancaster and Salkauskas [4]. Thus, MLS approximation is very suitable to form the shape function used in the meshless methods. At the domain Ω, the MLS approximation function (trial function), $u^h(\mathbf{x})$, for the displacement function $u(\mathbf{x})$ is expressed by the vector of the polynomial basis function, $\mathbf{p}^T(\mathbf{x})$, and the vector of coefficients, $\mathbf{a}(\mathbf{x})$, as follows

$$u^h(\mathbf{x}) = \sum_{j=1}^{m} p_j(\mathbf{x})\, a_j(\mathbf{x}) = \mathbf{p}^T(\mathbf{x})\, \mathbf{a}(\mathbf{x}), \tag{3}$$

where $p_j(\mathbf{x})$ is the monomial at the spatial coordinate $\mathbf{x}^T = \lfloor x, y \rfloor$, and m is the number of monomial in the basis function. The general form of the basis function with the degree s on two dimensional can be expressed by $\mathbf{p}^T(\mathbf{x}) = \lfloor 1, \ldots, x^s, x^{s-1} y, \ldots, xy^{s-1}, y^s \rfloor$. The term "moving" is used due to the dependability of coefficient $\mathbf{a}$ to the variable $\mathbf{x}$, whereas the weight function $w_I(\mathbf{x}) = w(\mathbf{x} - \mathbf{x}_I)$ is defined for every different evaluation point $\mathbf{x}$ at the domain.

Considering that the coefficient $\mathbf{a}$ in Equation 3 does not have a physical meaning, the approximation of displacement field values needs to be modified by stating coefficient $\mathbf{a}$ as *d.o.f.* at nodal $\mathbf{u}$, so that an equation similar to that of the interpolation function used in the finite element method can be obtained as

$$u^h(\mathbf{x}) = \boldsymbol{\phi}^T(\mathbf{x}) \,.\, \hat{\mathbf{u}} = \sum_{I=1}^{n} \phi_I(\mathbf{x})\, \hat{u}_I \,; \qquad u^h(\mathbf{x}_I) \equiv u_I \neq \hat{u}_I \tag{4}$$

where $\phi_I(\mathbf{x})$ denotes the shape function from the MLS approximation, where $\phi_I(\mathbf{x}) = 0$ if the weight function $w_I(\mathbf{x}) = 0$, and n is the number of nodes around the evaluation point $\mathbf{x}$, only in the areas where $w_I(\mathbf{x}) > 0$ (also known as the domain of definition of the point $\mathbf{x}$). The support of a weight function (also known as the domain of influence of a node) refers to the subdomain where the value of the weight function is not equal to zero. The illustration of the MLS approximation's shape function can be seen in Figure 1. It is important to understand that $\hat{u}_I$ is not the real nodal values of the unknown trial function $u^h(\mathbf{x})$, instead it represents the fictitious nodal values. The difference between u_I dan $\hat{u}_I$ on one dimensional problems can be described in Figure 2.

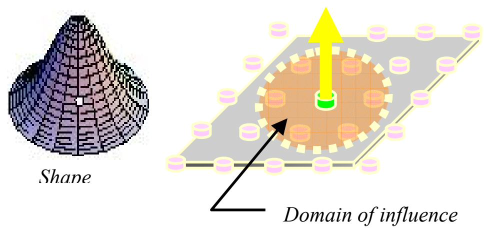


Figure 1. Illustration of the shape function used in meshless methods[5]

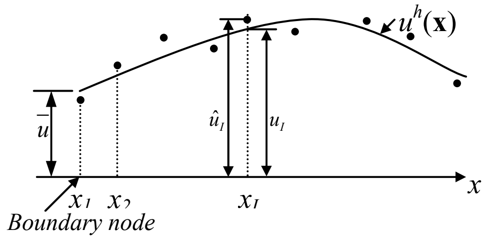


Figure 2. The difference between u_I dan $\hat{u}_I$ displacement[2]

3.2. ELEMENT-FREE GALERKIN (EFG)

EFG is first introduced by Belytschko et al. [1]. Although the definition of elements is no longer needed in EFG method, this method still cannot be considered as "truly meshless" because it requires the use of a background cell in the problem domain to execute the numerical integration procedure (Figure 3). The weak form of the governing equations are satisfied globally in this method. Thus, the numerical integration is done in the whole domain. This procedure requires a lot of quadrature points to be defined and placed in each of the background cells.

In EFG, the displacement field u_i acts as a trial function, and the virtual displacement field δv_i serves as a test function. Multiplying the test function to the both sides of the governing equation, followed by integrating on the domain Ω will give

$$\int_{\Omega} (\sigma_{ij,j} + F_i)\, \delta v_i \, d\Omega = 0. \tag{5}$$

By applying differentiation by parts, Gauss divergence theorem, and noting that $t_i = \sigma_{ij}.n_j$ (where $t_i = 0$ at Γ, except at the boundary Γ_t where $t_i = \bar{t}_i$), the following is obtained

$$\int_{\Gamma_t} \bar{t}_i\, \delta v_i \, d\Gamma + \int_{\Omega} F_i\, \delta v_i \, d\Omega - \int_{\Omega} \sigma_{ij}\, \delta v_{i,j}\, d\Omega = 0 \tag{6}$$

The shape function of the MLS approximation does not have the Kronecker delta property, thus $\phi_I(x_J) \neq \delta_{IJ}$, where $\phi_I(x_J)$ is the shape function of the nodal I evaluated on the nodal point x_J, and δ_{IJ} is the Kronecker delta (where $\delta_{IJ} = 1$ if $I = J$, and $\delta_{IJ} = 0$ if $I \neq J$). Due to this property, the trial function does not satisfy the essential boundary conditions, $u_i = \overline{u}_i$ at Γ_u. In EFG the Lagrange multiplier is introduced to overcome this problem, giving the equation

$$\int_{\Omega} \sigma_{ij}\delta v_{i,j}\, d\Omega - \int_{\Omega} F_i \delta v_i\, d\Omega - \int_{\Gamma_t} \overline{t}_i\, \delta v_i\, d\Gamma - \int_{\Gamma_u} (u_i - \overline{u}_i)\delta\lambda_i\, d\Gamma - \int_{\Gamma_u} \lambda_i \delta v_i\, d\Gamma = 0, \tag{7}$$

where λ_i and $\delta\lambda_i$ are the Lagrange multiplier and variation of Lagrange multiplier, respectively.

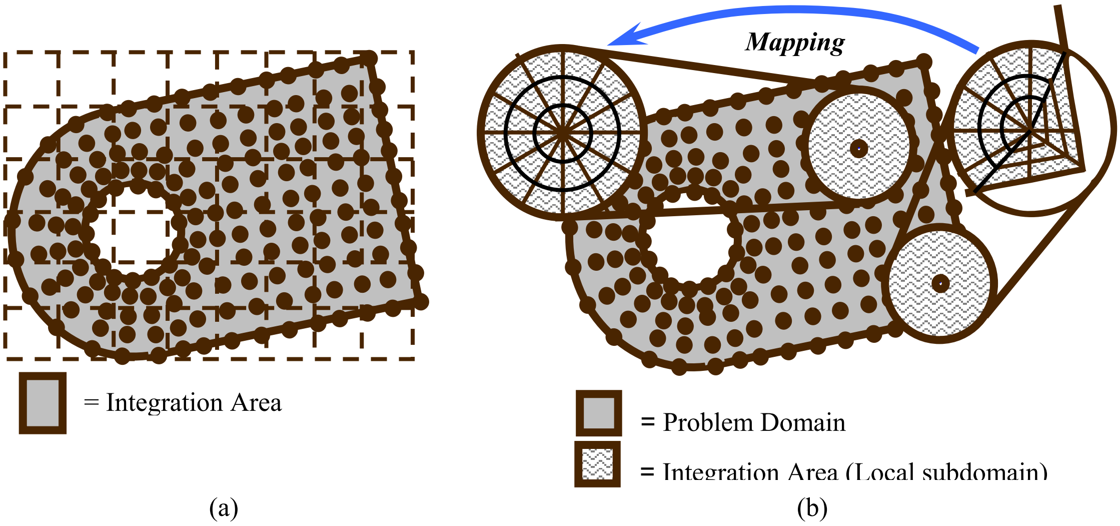


Figure 3. Numerical integration area (a) using the background cell over the entire problem domain in EFG[6], and (b) over the local subdomain in MLPG[6]

3.3. MESHLESS LOCAL PETROV-GALERKIN (MLPG)

The weak form on MLPG are not satisfied globally on the entire problem domain Ω, but it is done in the local subdomain (leading to the term Local Symmetric Weak Form) which resides entirely inside the global domain Ω, thus the MLPG can be considered as "truly meshless" (Figure 3). Moreover, the essential boundary conditions ($u_i = \overline{u}_i$ pada Γ_u) also cannot be satisfied directly in this method using the MLS approximation. MLPG solves this problem by introducing the use of a penalty function. MLPG is first introduced by Atluri et al. [2].

By multiplying the test function to both sides of the local weak form of the governing equations given in Equation 1 and the boundary conditions given in Equation 2 and introducing the penalty method, the following is obtained

$$\int_{\Omega_I^{te}} (\sigma_{ij,j} + b_i)\, v_i\, d\Omega - \alpha \int_{\Gamma_I^{su}} (u_i - \overline{\overline{u}}_i)\, v_i\, d\Gamma = 0, \tag{8}$$

where Γ_I^{su} is the intersection of Γ_u with the boundary $\partial\Omega_I^{te}$, u_i and v_i are the trial function and the test function respectively, and α is the penalty function. Using the differentiation by parts and the Gauss divergence theorem, and taking into account the non-essential boundary conditions, the following local symmetric weak form is obtained :

$$\int_{\partial\Omega_I^{te}} \sigma_{ij}\, v_{i,j}\, d\Omega + \alpha\int_{\Gamma_I^{su}} u_i\, v_i\, d\Gamma - \int_{\Gamma_I^{su}} t_i\, v_i\, d\Gamma = \int_{\Gamma_I^{st}} \bar{t}_i\, v_i\, d\Gamma + \alpha\int_{\Gamma_I^{su}} \bar{u}_i\, v_i\, d\Gamma + \int_{\Omega_I^{te}} F_i\, v_i\, d\Omega, \quad (9)$$

where $t_i = \sigma_{ij}.n_j$, and Γ_I^{st} is the intersection of Γ_t and boundary $\partial\Omega_I^{te}$.

3.4. FINITE POINT (FP)

Finite point method, proposed by Onate et al. [3], uses the the point collocation scheme to eliminate the integral appearing in the general weighted residual form.

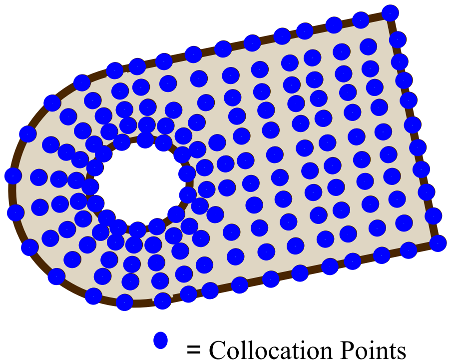


Figure 4. Collocation points where the governing differential equations are satisfied in Finite Point Method [6]

Weighted residual method is a common procedure to solve the governing differential equation numerically, where the displacement function u is approximated by a trial function u^h giving

$$\int_{\Omega} W_I\left[A\left(u_i^h\right)+F_i\right]d\Omega+\int_{\Gamma_t} \bar{W}_I\left[B\left(u_i^h\right)-\bar{t}_i\right]d\Gamma+\int_{\Gamma u} \bar{\bar{W}}_I\left[u_i^h-\bar{u}_i\right]d\Gamma=0, \quad (10)$$

where $W_I, \bar{W}_I$, dan $\bar{\bar{W}}_I$ denote the weight functions corresponding with Equation 1 and the boundary conditions in Equation 2 respectively. Point collocation is then used to eliminates the integral appearing in Equation 10 and the weight function is set to $W_I = \bar{W}_I = \bar{\bar{W}}_I = \delta\left(x_I - x\right)$, with the Dirac delta component as follows

$$\delta\left(x_I - x\right) = \begin{cases} 0, & x \neq x_I \\ 1, & x = x_I \end{cases} \quad (11)$$

The discrete equations in finite point method is obtained by substituting the approximation function of the displacement, u^h, to the governing differential equation stated in Equation 1 and boundary conditions in Equation 2, combined with the application of point collocation procedure. The discrete equations are given as

$$[\, A\left(u_i^h\right) + F_i\,]_p = 0 \qquad \text{at } \Omega, \qquad p = 1, 2,\ldots N_r, \quad (12a)$$

$$[\, \left(u_i^h\right)\,]_s = \bar{u}_i \qquad \text{at } \Gamma_u, \qquad s = 1, 2,\ldots N_u, \quad (12b)$$

$$[\, B\left(u_i^h\right) - \bar{t}_i\,]_r = 0 \qquad \text{at } \Gamma_t, \qquad r = 1, 2,\ldots N_t, \quad (12c)$$

where N_r is the number of nodal points on the domain Ω, except the ones at the boundary Γ_u and Γ_t, while N_u and N_t are the numbers of nodal points at the boundary Γ_u and Γ_t respectively. A and B symbolizes the differential operator to define the governing differential equations need to be satisfied at Ω dan Γ_t respectively, where $A\left(u_i^h\right) = \sigma_{ij,j}$ and $B\left(u_i^h\right) = \sigma_{ij} n_j$ [7].

4. TEST PROBLEM AND DISCUSSION

Consider an infinite plate with a central hole. The hole is a circle defined by $x^2 + y^2 \le a^2$, where a is the radius of the circle. The plate is subjected to a uniform tension, $\sigma = 1$, in the x direction, as shown in Figure 5a. The exact solutions for stresses, given by Timoshenko et al. [8], are

$$\sigma_x\left(x,y\right) = 1 - \frac{a^2}{r^2}\left\{\frac{3}{2}\cos 2\theta + \cos 4\theta\right\} + \frac{3a^4}{2r^4}\cos 4\theta, \tag{13a}$$

$$\sigma_y\left(x,y\right) = -\frac{a^2}{r^2}\left\{\frac{1}{2}\cos 2\theta - \cos 4\theta\right\} + \frac{3a^4}{2r^4}\cos 4\theta, \tag{13b}$$

$$\sigma_{xy}\left(x,y\right) = -\frac{a^2}{r^2}\left\{\frac{1}{2}\cos 2\theta + \sin 4\theta\right\} + \frac{3a^4}{2r^4}\cos 4\theta, \tag{13c}$$

where (r, θ) are the polar coordinates and θ is measured from the positive x axis counterclockwise.

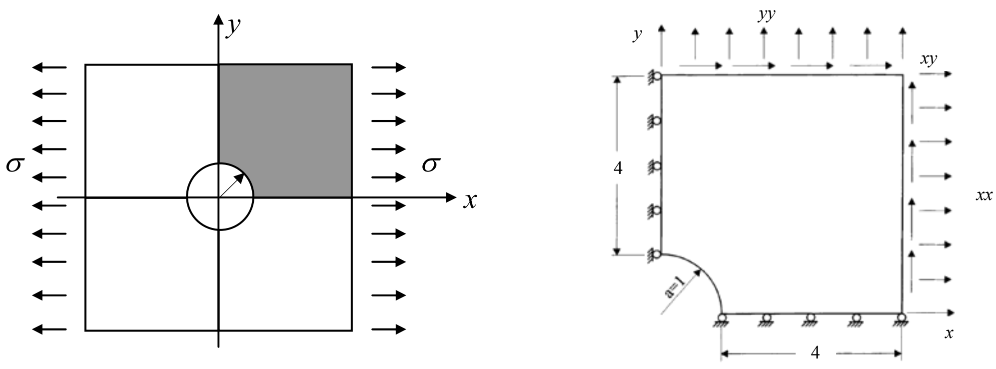
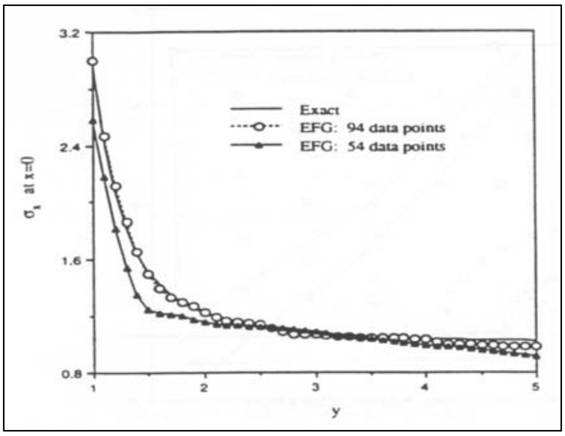


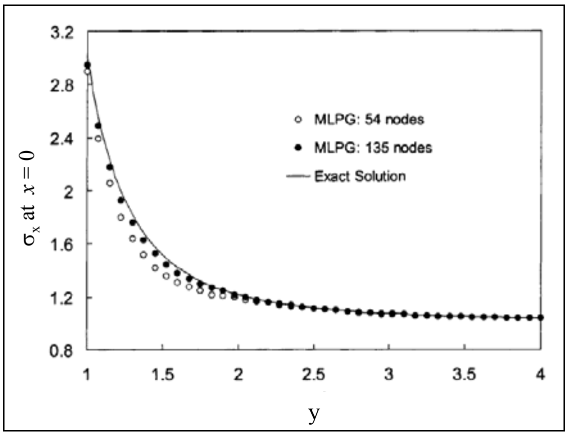
Figure 5. (a) Infinite plate with a center hole, and (b) its upper right part, with the traction at the outer boundary [9]

Due to symmetry, only a part of the upper right quadrant of the plate is modeled, as shown in Figure 5b. Symmetry conditions are imposed on the left ($u_x = 0$, $t_y = 0$) and bottom ($u_y = 0$, $t_x = 0$) edges, and the inner boundary at $a = 1$ is traction free. The non-essential boundary conditions given by the exact solution in Equation 13 are imposed on the right and top edges.

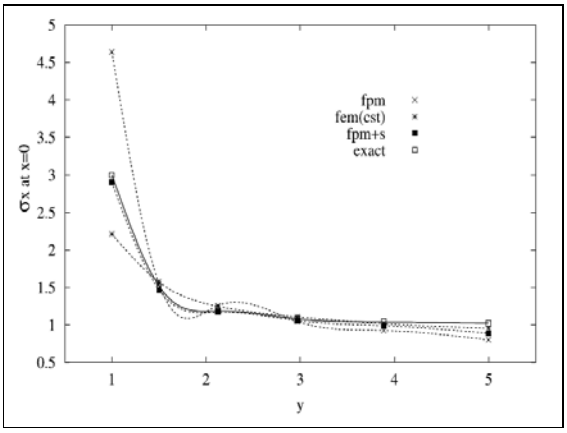
This test problem has been evaluated previously by using each of the three aforementioned meshless methods. Belytschko et al. [1], Atluri et al. [9], Onate et al. [7] evaluated the test problem using the EFG method, MLPG method, and FP method, respectively. Here, the results are presented and compared. The variable to be compared is the normal stress in the horizontal direction along the line x=0. The results from the three meshless methods will then be compared by the ones obtained by using the finite element method. The finite element method analysis is done by SAP2000 [10] program. The plotted stress results are presented in Figure 6.



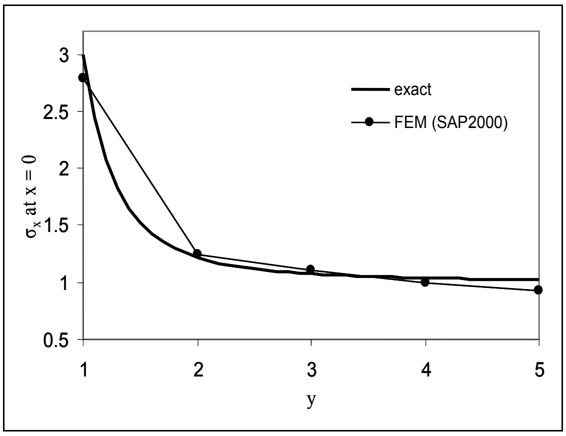
(a)



(b)



(c)



(d)

Figure 6. The plotted stress results from the plate with a center hole obtained by using (a) EFG method [1] (b) MLPG method [9] (c) FP method [7], and (d) Finite Element Method [5]

As seen in Figure 6, all methods used yield relatively good results compared to the exact solutions given in Equation 13. Results from EFG and MLPG methods also show the convergence property. In the results obtained by using finite element method, it can be seen that the plot is not as smooth as the ones obtained by using the meshless method. This is due to the fact that the shape function obtained from the MLS approximation itself is already smooth in geometry as shown in Figure 1, and its shape function derivatives also have smooth and continuous character although the basis function used may only be a linear one. The approximation function used in meshless methods also allows any points in the domain to be approximated by using more number of nodes than the unknown variables due to its least squares property. The finite element method used in SAP2000 uses the bi-linear basis function in its interpolation scheme. Due to its linear interpolation feature, the smoothness of the plotted results relies strongly on the number of nodes used. Table 1 provides a closer look at the numerical values obtained from the plotted results shown in Figure 6.

Table 1. σ_x at $x = 0$ for the plate with a center hole [5]

Method used	$y = 1$	$y = 2$	$y = 3$	$y = 4$	$y = 5$
Finite Element Method (54 nodes)	2.78	1.25	1.1	1	0.93
Element-free Galerkin (54 nodes)	2.63	1.21	1.06	1.01	0.97
Meshless Local Petrov-Galerkin (54 nodes)	2.96	1.22	1.05	1.02	1.01
Finite point (60 nodes)	2.75	1.24	1.13	1	0.91
Exact solution	3	1.23	1.07	1.04	1.02

5. CONCLUSIONS

Each meshless method has its own specific character and feature, but all of them emphasizes on the unnecessity to define elements in the discretization of the problem domain. EFG method, being the first well published meshless method, is not truly meshless in a sense that it still requires a background cell to evaluate the weak form. MLPG does not require a background cell and evaluates the weak form over intersecting local sub-domains, therefore it is "truly meshless". FP method has the advantage over the other methods since absolutely no integration scheme is needed in the weak form evaluation due to the use of point collocation scheme, but it has been considered that by using point collocation scheme, more nodes are required to achieve the same degree of correctness.

The test problem shows that meshless methods have the advantage over the finite element method due to the smoothness of the shape function and shape function derivatives used, aside from its obvious advantage that absolutely no elements need to be defined in the problem domain discretization. With the advantages and room for improvement, meshless methods deserve more research and attention as an alternative to the widely-known finite element method.

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