Stochastic Judgments in the AHP: Confidence Interval Construction Using Score Statistics Siana Halim† and Indriati N.

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Abstract. In multicriteria decision-making methods, such as the Analytic Hierarchy Process (AHP), single values are used to compare criteria and alternatives. Usually this single value given by decision makers follows the fundamental scale from 1 to 9. However, a decision maker often does not have complete support information for his or her making decisions. This lack of information causes the decision...
The decision maker provides a single-valued pairwise preference judgments, yielding a $k \times k$ matrix $A = \{a_{ij}\}$ of preference ratios with respect to a given criterion $C$, where $k$ is the number of evaluated alternatives.

If the relative preference statements are represented by judgment intervals, rather than single values, then the rankings resulting from a classic (deterministic) AHP analysis based on single judgment values may be reversed, and therefore incorrect.

Stam and Silva (1997) developed statistical techniques to obtain both point estimates and confidence intervals of the rank reversal probabilities.

They also simulated the realization of $a_{ij}$ uniformly with allowing inconsistencies between the pairwise comparisons. In this paper we constructed confidence interval following Stam and Silva (1997). The simulation was modified to avoid inconsistencies in the pairwise comparisons using the consistency improving method (Xu, et al., 1997) and setting a margin such that the improved values will be out of range (Rahardjo, et al., 2001). In addition, instead of using the Clopper-Pearson statistics (1934), which was used by Stam, we proposed to use the score statistic. It is well known that the coverage probabilities of Clopper-Pearson is too high and the score statistics behaves well (Agresti, 2002).
The Confidence Interval of the Probability Rank Reversal Construction. The construction of the confidence interval of probability of rank reversal \( \Pi_{ij} \) between alternatives i and j will follow Stam and Silva (1997) approached. The calculation of the rank reversal will be in the same lines as in the classical AHP methodology, therefore we need information about the true principal right eigenvector \( w = (w_1, \ldots, w_k) \)^T associated with interval judgments. Denote the pairwise comparison of alternatives i and j (i, j = 1, ..., k) by \( m_{ij} \), and let \( M = \{m_{ij}\} \). We simulated the realization \( a_{ij} \) for each entry of \( M \) above for \( i < j \), and set \( a_{ij} = 1/a_{ij} \) for \( i > j \), completing the reciprocal matrix A. We checked the inconsistency and modified the simulation using consistency improving method when it occurred during the simulation. For each generated A, we calculated the principal right eigenvector \( w \). Replicating this simulation \( n \) times, we obtain a sample \( w_1, \ldots, w_n \) principal eigenvectors.

Rank reversal between two alternatives i and j occurs when alternative i is preferred over j under perfect information (i.e. \( i > j \)), but it is calculated to be less preferred based on the sample information on the interval judgments (i.e. \( w_i < w_j \)). Let

\[
\pi_{ij} = P(i > j) \quad \text{and} \quad \pi_{i1j} = P(w_i < w_j),
\]

then \( \Pi_{ij} = \pi_{ij} (1 - \pi_{i1j}) + (1 - \pi_{ij}) \pi_{i1j} \). If we assume that approximately equal to the probability of \( (i > j) \) under complete information than equation (1) can be estimated as \( \Pi'_{ij} = 2\pi_{ij}(1 - \pi_{ij}) \) (2) It can be seen clearly, that (2) following the binomial distribution, \( \Pi'_{ij} \sim \text{bin}(2,1) \). The score confidence interval contains \( \Pi_0 \) values for which \( z_\alpha < z_\alpha / 2 \). Its endpoints are the \( \Pi_0 \) solutions to the equations

\[
\Pi_0 (1 - \Pi_0) / n \leq -\Pi_0 \leq \Pi_0 (1 - \Pi_0) / n + \pm z_\alpha / 2
\]

These are quadratic in \( \Pi_0 \). Firstly discussed by E.B. Wilson (1927), the interval is \( \Pi_0 = n / n + 2 \) \( n + z_\alpha / 2 \) \( n + z_\alpha / 2 + 1 \) \( z_\alpha / 2 \) \( n + z_\alpha / 2 \).
2 (4) za / 2 1 The midpoint $\Pi$ of the interval is a weighted average of $\Pi^{*}$ and $1/2$, where the weight $n/(n + za2 / 2)$ given $\Pi^{*}$ increases as $n$ increases. Stam and Silva used the Cooper and Pearson Confidence interval (1934) as follows, $P_{ijL} = pij \pi_{ij} + (1 - pij + n-1)F_{a/2,2n(1-pij+n-1)}; 2npij pi_{Uj} = 1 - pij 1 - pij + ( pij +n-1)F_{a / 2,2n( pij +n- )}; 2n(1 - pij )$ (5).2.2 Consistency Improving Method (CIM) Let $R_{k^+} = \{x = (x_1, x_2, ..., x_k)r | x_i > 0, i = 1, 2, ..., k\}$ 

0, i =1,2, ..., k) Lemma 1 Let $A = (aij)$ is an $k \times k$ positive matrix and $\lambda_{\text{max}}$ is the maximum eigenvalue of $A$. Then

$\lambda_{\text{max}} = \min \max \sum aij n \times j \times R_{n+ i j} = 1 \times i$ (6)

Let $A$ and $\lambda_{\text{max}}$ as in Lemma 1. The positive right eigenvector with respect to $\lambda_{\text{max}}$ is called as the principal right eigenvector of $A$. Lemma 2. Let $x > 0, y > 0, \lambda > 0$ and $\mu > 0$, and $\lambda + \mu = 1$. Then $x \lambda y \mu \leq \lambda x + \mu y$. The equality is reached if and only if $x = y$ Lemma

3 Let $A$ is an $k \times k$ positive reciprocal matrix, $\lambda_{\text{max}}$ is the maximum eigenvalue of $A$. Then $\lambda_{\text{max}} \geq k$. The equality is reached if and only $A$ is consistent. Theorem 1. Let $A = (aij)$ is a $k \times k$

positive reciprocal matrix, and $\lambda_{\text{max}}$ is the maximum eigenvalue of $A$, $w = (w_1, w_2, ..., w_n)T$ is the principal right eigenvector of $A$. Let $B = (bij)$, where $bij = (aij)\lambda wi 1-\lambda (7) w j$

Let $\mu_{\text{max}}$ is the maximum eigenvalue of $B$ then $\mu_{\text{max}} \leq \lambda_{\text{max}}$, the equality is reached if and only if $A$ is consistent.
Proofs of lemmas and theorems above can be seen at Xu and Wei (1999). Through Theorem 1, the inconsistent matrices can be transformed into consistent matrices by, \( a_{ij}(j^{k+1}) = (a_{ij}(j^{k})) \lambda w_{wi}(j^{k}) \).

1-\( \lambda \) (8) In this transformed matrix, the consistency criteria are altered as follows: \( \delta = \max \{|aim - a_{ij}o|\} \), \( i, j = 1, 2, ..., k \). 

\[
i, j = 1, 2, ..., k \quad \sum_{i=1}^{k} \sum_{j=1}^{k} (a_{ij}(j^{n}) - a_{ij}(j^{0}))^2 \sigma = k (9)\]

where \( \delta < 2 \) and \( \sigma > 1 \). CIM is valid if the consistency ratio less than 0.1. However, Raharjo et al. (2001) showed that CIM has two disadvantages. First, there is a possibility that the result of CIM lies outside the fundamental scale of AHP. Moreover, in one case study, 33.67% of resurvey results showed different results from the CIM. In the simulation these two disadvantages can be neglected. We only need to pay more attention to the first one, that is, by generating more random matrix until the consistency fulfilled and the range of each matrix elements is inside the fundamental scale.

3. NUMERICAL EXAMPLE

Suppose a decision maker decides to use AHP for comparing four alternatives A1,…, A4. We simulated random uniform number between 1-9 and between 1/9-1 as the element of the comparison matrices. If these matrices are not consistent then we modified using the modified CIM until the inconsistencies in the matrices are solved. Then we normalized the matrices using geometric mean to get \( P_{ij} \). We used this relationship for calculating \( P_{ij} \) as follows, if \( A = 4B \) then \( P_{ij} = A = 4B \).

\[
\begin{align*}
\text{Table 1: Comparing Confidence Interval of pairwise preferences using Fisher and Score statistics} \\
&\text{Pair Pij | [Pij_L, Pij_U] | Phi [Pij_L, Pij_U]} \\
&\phi_1 \phi_2 \phi_3 \phi_4 \phi_5 \phi_6 \phi_7 \phi_8 \phi_9 \\
\end{align*}
\]

We constructed the confidence interval of \( P_{ij} \) using (4) Table 1. Comparing Confidence Interval of pairwise preferences using Fisher and Score statistics Pair Pij | [Pij_L, Pij_U] | Phi [Pij_L, Pij_U] (i,j) Fisher Fisher Scoring Scoring (1,1) 0.5 [0.0676,0.9324] 0.5 [0.1295,0.7123] (1,2) 0.5394 [0.0833,0.9465] 0.521 [0.1833,0.7565] (1,3) 0.5071 [0.0703,0.9351] 0.572 [0.1703,0.7351] (1,4) 0.4855 [0.0622,0.9268] 0.518 [0.1622,0.7268] (2,1) 0.4606 [0.0535,0.9167] 0.546 [0.1535,0.7167] (2,2) (2,3) (2,4) (3,1) (3,2) (3,3) (3,4) (4,1) (4,2) (4,3) (4,4) 0.5 [0.0676,0.9324] 0.500 [0.1676,0.7324] 0.5463 [0.0862,0.9488] 0.573 [0.1862,0.7488] 0.5148 [0.0733,0.9379] 0.582 [0.1733,0.7379] 0.4929 [0.0649,0.9297] 0.529 [0.1649,0.7979] 0.4537 [0.0512,0.9138] 0.535 [0.1512,0.718] 0.5 [0.0676,0.9324] 0.5 [0.1676,0.724] 0.4526 [0.0508,0.9133] 0.526 [0.1508,0.713] 0.5145 [0.0732,0.9378] 0.515 [0.1732,0.7778] 0.4852 [0.0621,0.9267] 0.522 [0.1621,0.7272] 0.5474 [0.0867,0.9492] 0.511 [0.1867,0.7472] 0.5 [0.0676,0.9324] 0.5 [0.1676,0.732] Table 1 shows that \( P_{ij} \) lies in between 0.5, this is true since we generated the elements of the matrices from uniform distribution. Hence, the preferences probability are equal for every alternatives. Moreover, the confidence intervals constructed via score statistics show they are narrower than ones constructed via the Pearson-Copper Statistics. Therefore, we can conclude that the score statistics are more robust than the Pearson-Copper for this case.

4. CONCLUSION

In this paper we constructed the confidence interval for preferences judgment using score statistics. We simulated the data by generated the element of the matrices uniformly and checked the consistency index using modified consistency index method. The result shows the nature of the uniformly data, that is, the equality of preferences in every alternatives.

REFERENCES