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Development of the DKMQ element for buckling analysis of shear-deformable plate bending

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Abstract

In this paper the discrete-Kirchhoff Mindlin quadrilateral (DKMQ) element was developed for buckling analysis of plate bending including the shear deformation. In this development the potential energy corresponding to membrane stresses was incorporated in the Hu-Washizu functional. The bilinear approximations for the deflection and normal rotations were used for the membrane stress term in the functional, while the approximations for the remaining terms remain the same as in static analysis. Numerical tests showed that the element has good predictive capability both for thin and thick plates.
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1. Introduction

Plate bending is of an utmost important structural model in engineering. To analyze practical problems of plate bending, the finite element method (FEM) is at present the most widely used numerical method. Indeed the plate bending problem is one of the earliest problems to which the FEM was applied [1]. The most commonly used theories in developing finite elements for analysis of plate bending are Kirchhoff (or thin plate) and Reissner-Mindlin (or thick plate) theories. The Kirchhoff plate theory neglects the effect of shear deformation and thus it is only valid for thin plates, whereas the Reissner-Mindlin (RM) plate theory is applicable to both thick and thin plates. In early development of the FEM, the Kirchhoff theory was widely adopted as the basis of the finite element

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formulation. The difficulty with this approach is to construct the shape functions that satisfy the C^1 continuity requirement. In the subsequent developments the RM theory is preferred since it requires only C^0 continuity on the shape functions and furthermore, it is a more general theory than the Kirchhoff theory.

While the use of RM theory in developing plate elements by-passes the difficulty caused by the C^1 requirement, direct application of the displacement-based finite element formulation, however, produces elements that overly stiff for thin plate situations. This phenomenon is known as shear locking. Early attempts to overcome this difficulty wasto employ the selective reduced integration technique (e.g. see [2] and the references therein). Unfortunately this simple approach produced elements that have spurious energy modes. Since then there are innumerable RM plate bending elements have been proposed with different approaches to eliminate the shear locking. Some recently proposed successful plate bending elements include the refined Mindlin plate elements [3,4], a family of RM plate elements formulated using the discrete shear gap concept [5,6], and the RM plate element based on the consistent version of the Mindlin equations [7].

Among countless plate bending elements available now, the discrete-Kirchhoff Mindlin quadrilateral (DKMQ) element proposed by Katili [8] is of our interest since it has the standard nodal degrees of freedom, pass the patch test, shear locking free, and no spurious zero energy modes. Furthermore it has been proven [8] that the DKMQ has good predictive capability for thin to thick plates. This element is an extension of the DKQ (discrete Kirchhoff quadrilateral) element [9], which is a simple, efficient and reliable element for analysis of thin plates to include the shear deformation. The DKMQ [8] results will converge to the DKQ [9] results as the plate becomes progressively thinner.

With regard to the good characteristics of the DKMQ element, this element has been recently further developed to the DKMQ24 shell element [10] and applied to composite plate bending structures [11,12]. However, to the authors' knowledge, there is no published report on the application of the DKMQ to plate bending buckling problems. It is thus the aim of this paper to present the development of the DKMQ element to plate buckling problems.

In the present development the membrane strain energy was added to the original Hu-Washizu functional for RM plates in order to account for the membrane stress effect to the plate bending stiffness. The approximate deflection and rotation fields for the membrane strain energy were taken to be the standard bilinear function, while the approximate fields for the bending and shear strain energy followed the original work [8]. The element was tested to different plate buckling problems to assess the accuracy and convergence characteristics. The results showed the DKMQ element can give accurate critical bucking loads both for thin and thick plates.

2. Formulation of the DKMQ for bucking analysis

A detailed formulation of the DKMQ for static analysis of plate bending have been presented in Reference [8]. In this section we present only the essential equations of the static formulation. The focus is given to formulation of the DKMQ for buckling problems.

2.1. Variational formulation

We consider a plate of uniform thickness h, made from homogeneous and isotropic material with modulus of elasticity E and Poisson's ratio ν. Three dimensional Cartesian coordinate system is established with the x-y plane lying on the plate middle surface A as illustrated in Fig 1. Based on basic assumptions of the RM plate theory, the displacement of a generic point in the plate can be expressed as

$$u = z\beta_x(x,y), \quad v = z\beta_y(x,y), \quad w = w(x,y) \tag{1}$$

Where w is the deflection of the middle surface A, β_x and β_y are the normal line rotations in the x-z and y-z planes, respectively.

The strains associated with bending deformation, $\langle\varepsilon_b\rangle$, can be expressed in terms of the curvature, $\langle\chi\rangle$, as

$$\langle\varepsilon_b\rangle = z\langle\chi\rangle, \;\; \langle\chi\rangle = \langle\frac{\partial\beta_x}{\partial x}\frac{\partial\beta_y}{\partial y} \;\; (\frac{\partial\beta_x}{\partial y}+\frac{\partial\beta_y}{\partial x})\rangle \tag{2a, b}$$

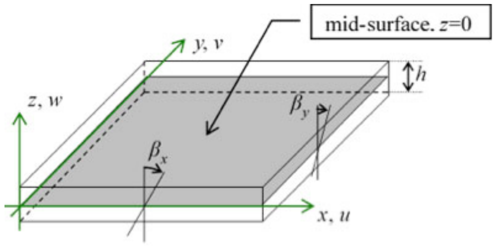


Fig. 1. Plate with the coordinate system and positive sign convention for the displacement fields.

The strains associated with transverse shear deformation are given as

$$\langle\gamma\rangle = \langle\gamma_x\gamma_y\rangle = \langle\frac{\partial w}{\partial x} + \beta_x \frac{\partial w}{\partial y} + \beta_y\rangle \tag{3}$$

Here and here after symbols $\langle\circ\rangle$, $\{\circ\}$, and $[\circ]$ are used to denote a row matrix, a column matrix, and a square matrix, respectively.

Following the static formulation[8], the Hu-Washizu functional is used as the basis to develop the DKMQ element for plate buckling analysis. In the presence of membrane (or in-plane) pre-buckling stressesσ_x^0, σ_y^0 and τ_{xy}^0 and in the absence of other external forces, the Hu-Washizu functional is given as[8], [13-16]

$$\Pi = \Pi_b(\beta_x, \beta_y) + \Pi_s\left(w, \beta_x, \beta_y, \{\underline{\gamma}\}, \{T\}\right) + \Pi_\sigma(w, \beta_x, \beta_y) \tag{4}$$

Where

$$\Pi_b = \frac{1}{2}\int_A \langle\chi\rangle\,[H_b]\{\chi\}dA \tag{5}$$

$$\Pi_s = \frac{1}{2}\int_A \langle\underline{\gamma}\rangle\,[H_s]\{\underline{\gamma}\}dA + \int_A \langle T\rangle\left(\{\gamma\} - \{\underline{\gamma}\}\right)dA \tag{6}$$

$$\Pi_\sigma = \frac{1}{2}\int_A \langle\nabla w\rangle\,[\sigma_0]\{\nabla w\}h dA + \frac{1}{2}\int_A \langle\nabla\beta_x\rangle\,[\sigma_0]\{\nabla\beta_x\}\frac{h^3}{12}dA + \frac{1}{2}\int_A \langle\nabla\beta_y\rangle\,[\sigma_0]\{\nabla\beta_y\}\frac{h^3}{12}dA \tag{7}$$

In Eqns. (4)-(7)Π_b is the bending strain energy, $\Pi_s$is the shear strain energy, and Π_σ is the membrane strain energy associated with the buckling deformation. Matrix $\{\underline{\gamma}\}$ is a matrix of assumed shear strains and matrix $\{T\}$ is a matrix of the shear forces. Matrices $[H_b]$ and $[H_s]$ are the bending and shear elasticity matrices, respectively, given as

$$[H_b] = D_b\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}, \quad D_b = \frac{Eh^3}{12(1-\nu^2)} \tag{8}$$

$$[H_s] = D_s\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad D_s = kGh \tag{9}$$

Here D_b is the bending rigidity, D_s is the shear rigidity, k is the shear correction factor, which is taken to be 5/6, and G is the shear modulus, $G = E/2(1+\nu)$.

The expression of Π_σ, Eqn. (7), comes from the work of membrane stresses σ_x^0, σ_y^0 and τ_{xy}^0along the nonlinear terms of membrane Green strains[14,16]. In this equation, $[\sigma_0]$ is the matrix of membrane stresses, that is

$$[\sigma_0] = \begin{bmatrix}\sigma_x^0 & \tau_{xy}^0\\ \tau_{xy}^0 & \sigma_y^0\end{bmatrix} \tag{10}$$

Matrices $\langle \nabla w \rangle$, $\langle \nabla \beta_x \rangle$ and $\langle \nabla \beta_y \rangle$ are matrices of gradient of the deflection and rotations, that is,

$$\langle \nabla w \rangle = \langle \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \rangle \langle \nabla \beta_x \rangle = \langle \frac{\partial \beta_x}{\partial x} \frac{\partial \beta_x}{\partial y} \rangle \langle \nabla \beta_y \rangle = \langle \frac{\partial \beta_y}{\partial x} \frac{\partial \beta_y}{\partial y} \rangle \quad (11)$$

The stationary condition of Π with respect to $\{T\}$ gives[8,15]

$$\int_A \langle \delta T \rangle \left(\{\gamma\} - \{\underline{\gamma}\} \right) dA = 0 \quad (12)$$

Which is a constraint equation relating the assumed shear strains $\{\underline{\gamma}\}$ to the kinematical shear strains $\{\gamma\}$.

2.2. Approximation form

We consider a typical DKMQ element as shown in Fig. 2. The element has four nodes and three degrees of freedom per node, that is, w_i, β_{xi}, β_{yi}, i=1,… , 4. Mid-side nodes 5, …, 8 are used to define a nodal parameter $\Delta\beta_{sk}$, k=5,… , 8, which is the difference between the linearly and quadratically interpolated tangential rotation at a mid-side node. For the bending and shear strain energies Π_b and Π_s, approximate rotations within an element are given as [8]

$$\beta_x = \sum_{i=1}^{4} N_i \beta_{xi} + \sum_{k=5}^{8} P_k C_k \Delta\beta_{sk} \quad (13)$$

$$\beta_y = \sum_{i=1}^{4} N_i \beta_{yi} + \sum_{k=5}^{8} P_k S_k \Delta\beta_{sk} \quad (14)$$

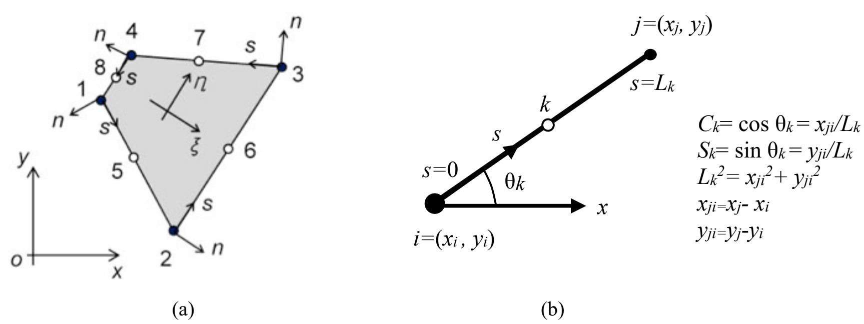
Where $N_i = \frac{1}{4}(1 \pm \xi)(1 \pm \eta)$, i=1,…, 4, are the bilinear shape functions and P_k, k=5,…, 8 are the hierarchical quadratic shape functions, all of them are expressed in terms of natural coordinates ξ-η. C_k and S_k are the direction cosines of side k. Approximate deflection and rotations for the membrane strain energy Π_σ, however, are given as

$$w = \sum_{i=1}^{4} N_i w_i \, \beta_x = \sum_{i=1}^{4} N_i \beta_{xi} \, \beta_y = \sum_{i=1}^{4} N_i \beta_{yi} \quad (15)$$

The assumed shear strains $\{\underline{\gamma}\}$are linearly interpolated from the discrete tangential shear strains at the mid-side nodes 5, …, 8. These mid-side tangential strains are obtained based on the moment-shear equilibrium equation along each side. The constraint equation, Eqn. (12), is implemented in discrete manner along each side using [8]

$$\int_0^{L_k} \left(\gamma_s - \underline{\gamma_s} \right) ds = 0, \;\; k = 5,6,7,8 \gamma_s = \frac{\partial w}{\partial s} + \beta_s \quad (16)$$

Where s is the tangential coordinate alongside k (see Fig. 2).



(a) (b)

Fig. 2. (a) DKMQ element with natural coordinates ξ-η and normal-tangential coordinates n-s along each side; (b) side k—nodes i, j (Taken from Reference [8]).

2.3. Geometric stiffness matrix

Substituting the approximate deflection and rotations, Eqns. (15), into Eqns. (11), followed by substituting the resulting gradient matrices into Eqn. (7), the membrane strain energy of an element can be expressed as

$$\Pi_\sigma = \frac{1}{2}\langle u_n\rangle[k_G]\{u_n\} \tag{17}$$

in which

$$\langle u_n\rangle = \langle w_1 \quad \beta_{x1} \quad \beta_{y1} \quad w_2 \quad \beta_{x2} \quad \beta_{y2} \quad w_3 \quad \beta_{x3} \quad \beta_{y3} \quad w_4 \quad \beta_{x4} \quad \beta_{y4}\rangle \tag{18}$$

is the nodal displacement vector and

$$[k_G] = h\int_A [G_w]^T[\sigma_0][G_w]\,dA + \frac{h^3}{12}\int_A \left[G_{\beta x}\right]^T[\sigma_0]\left[G_{\beta x}\right]dA + \frac{h^3}{12}\int_A \left[G_{\beta y}\right]^T[\sigma_0]\left[G_{\beta y}\right]dA \tag{19}$$

is the geometric stiffness matrix. In this equation

$$[G_w] = \begin{bmatrix} \dots & N_{i,x} & 0 & 0 & \dots i = 1,2,3,4 \\ & N_{i,y} & 0 & 0 & \end{bmatrix} \tag{20a}$$

$$\left[G_{\beta x}\right] = \begin{bmatrix} \dots & 0 & N_{i,x} & 0 & \dots i = 1,2,3,4 \\ & 0 & N_{i,y} & 0 & \end{bmatrix}\left[G_{\beta y}\right] = \begin{bmatrix} \dots & 0 & 0 & N_{i,x} & \dots i = 1,2,3,4 \\ & 0 & 0 & N_{i,y} & \end{bmatrix} \tag{20b, c}$$

where

$N_{i,x} = j_{11}N_{i,\xi} + j_{12}N_{i,\eta}N_{i,y} = j_{21}N_{i,\xi} + j_{22}N_{i,\eta}$(21)

And here$j_{11}, j_{12}, j_{21}, j_{22}$ are the components of the invers Jacobian matrix.

The reader may consult Reference [8] for a detailed formulation leading to the bending and shear stiffness matrices.

3. Numerical tests

In the following tests, we perform a series of plate buckling analyses to assess the convergence characteristics and accuracy of the developed DKMQ element in predicting a critical buckling load. The results are presented in terms of a non-dimensional buckling load intensity factor defined as $k_{cr} = L^2N_{cr}/\pi^2D_b$[13,17], where N_{cr} is the critical buckling load and D_b is the bending rigidity, as defined in Eqn. (8).

3.1. Simply-supported square plates subjected to an in-plane compressive load

We firstly consider a hard type simply-supported square plate with two different length-to-thickness ratios, that is L/h=10 and L/h=100, subjected to a uniaxial compressive load in x-direction. The length of the plate is L=10 m; the material properties are $E = 200\times10^9\,N/m^2$ and ν=0.3. The plate is modeled using different degrees of mesh refinement, i.e. 2×2, 4×4, 8×8, and 16×16.

The resulting buckling load intensity factors are presented in Table 1together with the analytical solutions. It is observed that the DKMQ results converge from above to the corresponding analytical solutions, both for the case of moderately-thick plate, L/h=10, as well as for the case of thin plate, L/h=100. 'Converging from above' is reasonable since the formulation basis is not a purely displacement-based formulation.

Table 1. Buckling load intensity factors for hard type simply-supported square plates.

Mesh	L/h=10	L/h=100
2x2	4.694	5.016
4x4	3.986	4.244
8x8	3.801	4.058
16x16	3.750	4.013
Analytical solution	3.741 ([18] as cited in[17])	4 [19]

3.2. Rectangular plates subjected to an in-plane compressive load

Secondly we consider hard type simply-supported rectangular plates of the length a and width b with five length-to-width ratios, a/b=0.5, 1, 1.5, 2, 2.5, and three thickness-to-width ratios, h/b=0.05, 0.1, 0.2, as described in Reference [13]. The plate is subjected to an in-plane compressive force N along the edges of width b. The material properties are the same as in the previous test. The plates are modeled using meshes with the number of elements along the edges of width b, n=12, while the number of elements along the edges of length a follows the ratio of a/b (thus the shape of all elements is square of the length $b/12$).

The resulting buckling load intensity factors are tabulated in Table 2 together with those obtained using the mesh free method with regular 17×17 particles presented by *Liew et al.*[13] and the pb-2 Ritz method presented by *Kitipornchai et al.* [20]. It is observed that results from the present element are in agreement with those obtained using the meshfree and pb-2 Ritz methods. For the case of thin plates(h/b=0.05), the results are a little bit higher compared to the reference results. As the plates become thicker (h/b=0.1 and h/b=0.2), however, the results are between the meshfree and pb-2 Ritz results.

Buckling mode shapes for the plates with length-to-width ratios a/b=2 and a/b=2.5 are shown in Fig. 3. These bucking modes are in agreement with those presented in Reference [13].

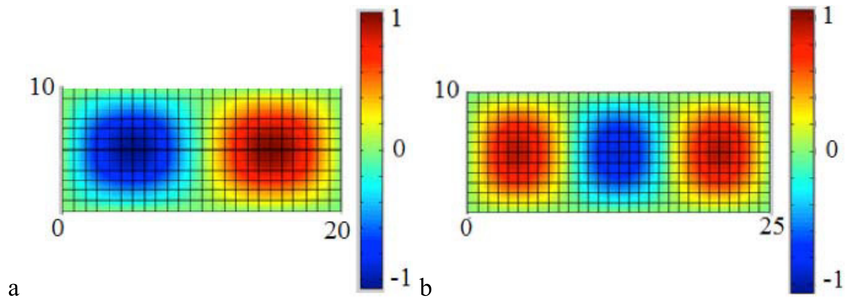


Fig. 3. Buckling mode shape for hard type simply-supported rectangular plates with length-to-width ratios: (a) a/b=2; (b) a/b=2.5.

Table 2. Buckling load intensity factors for hard type simply-supported rectangular plates with different length-to-width ratios a/band thickness-to-width ratios h/b.

a/b	h/b	Present	Meshfree [13]	pb-2 Ritz ([20] as cited in [13])
0.5	0.05	6.0967	6.0405	6.0372
	0.1	5.4085	5.3116	5.4777
	0.2	3.7877	3.7157	3.9963
1	0.05	3.9609	3.9293	3.9444
	0.1	3.7637	3.7270	3.7865
	0.2	3.1501	3.1471	3.2637
1.5	0.05	4.2764	4.2116	4.257
	0.1	3.9886	3.8982	4.025
	0.2	3.1638	3.1032	3.3048
2	0.05	3.9609	3.8657	3.9444
	0.1	3.7637	3.6797	3.7865
	0.2	3.1501	3.0783	3.2637
2.5	0.05	4.0817	3.9600	4.0645
	0.1	3.8376	3.7311	3.8683
	0.2	3.1134	3.0306	3.2421

3.3. A square plate with a hole subjected to different in-plane loads

Lastly we consider a hard-type simply-supported square plate with a hole of the thickness h=1, as shown in Fig. 4(a). The material properties are taken to be the same as in the first test (Sec. 3.1). The plate is subjected to three different in-plane load cases, namely an axial compressive load in x-direction, N_x, pure shear load, and biaxial compressive loads in x and y directions where $N_x = N_y$. The plate is modelled using a mesh as shown in Fig. 4(b).

The analysis results are tabulated in Table 3 and compared to those obtained using the element-free Galerkin method (EFGM) [17]. It can be seen that the results are very close to those of the EFGM, that is, around 1-3%higherthan the EFGM results.

Table 3. Buckling load intensity factors for the hard type simply-supported square plate with a hole.

Method	Axial	Shear	Biaxial
Present	2.041	7.978	1.059
EFGM [17]	1.986	7.867	1.032

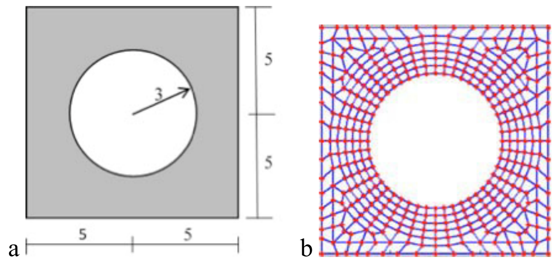


Fig. 4. (a) Hard type simply-supported square plate with a hole (adopted from [17], p. 474) and (b) its finite element mesh.

4. Conclusions

The DKMQ element has been extended for buckling analysis of plate bending. The static part of the formulation was taken to be the same as in the original formulation, while the geometric stiffness matrix was formulated based upon the standard bilinear approximation of the deflection and rotations in the membrane strain energy expression. The convergence and accuracy of the present formulation were tested using different plate buckling problems. The results showed that the element can yield accurate solutions both for thin and thick plates. The results converged from above as the finite element mesh was refined. Therefore, the DKMQ element can be very useful for predicting a critical buckling load in practical applications.

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