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### Articles

# Spatial Bayesian Model for Maximum Temperature

**Indriati N. Bisono**[1] **and Andrew P. Robinson**[2]

[1]Industrial Engineering Department
Petra Christian University
Jl. Siwalankerto 121-131, Surabaya 60236, Indonesia
mlindri@petra.ac.id

[2]School of Mathematics and Statistics
University of Melbourne, VIC 3010 Australia

**ABSTRACT**

*A three-stage Bayesian spatial model is fitted to temperature extremes covering Tasmania. In the first stage, the data in each grid cell are assumed to follow a GEV distribution with particular parameters. In the second stage, each GEV parameter is assumed to follow a Normal distribution with mean structure comprising a fixed and random effect component. A usual regression model with covariates longitude, latitude and elevation is employed for the fixed effect component, and a conditional auto regressive (CAR) model is used for the random effect. The estimation of the posterior parameters was conducted by Monte Carlo method using a hybrid MCMC of Metropolis and Gibbs sampler algorithm. We found the spatial random effect successfully smoothed the shape parameters, so that credible intervals of return levels were well behaved.*

**Keywords:** Bayesian, spatial, hierarchical model; CAR; extreme values.

**2000 Mathematics Subject Classification:** 91B72, 62F15, 62G32.

## 1 Introduction

According to the Technical Report for Climate Futures for Tasmania (**?**), a climate event is extreme when it occurs with greater intensity, frequency or duration than is normally expected. While there are many definitions of extremes, here we focus on maximum (or minimum) values over a period of time. There are several issues with extreme data analysis. Often only a small amount of data is available. For example, we might have daily temperature data for 20 years, and if we extract only the annual maximum in each year, we will have only 20 observations.
Spatial dependence occurs in most atmospheric data, and hence it is important to account for the spatial correlation when modeling such data. The spatial dependence can be captured by directly modeling the dependence at the data level or partially, by smoothing the parameters of the assumed distribution in the process level (Cooley, Nychka and Naveau, 2007; Hrafnkelsson, Morris and Baladandayuthapani, 2012; Kang and Cressie, 2011; Sang and Gelfand, 2009; Schliep, Cooley, Sain and Hoeting, 2010). The former is usually very difficult, and the latter arguably has the benefit of being more sensible and flexible in that the spatial association

is introduced as an adjustment to the explained covariates, usually through random effects (Banerjee, Carlin and Gelfand, 2004).

In this paper, we model temperature extremes generated by a regional climate model over Tasmania by using the Bayesian hierarchical approach of Schliep et al. 2010. The model is summarized in the next section, and then its implementation and some results are described. We show the generated future observations from the posterior predictive distribution, the credible interval of return levels from both the spatial and independent cell-wise model, and close with a summary.

## 2 The Three-Stage Model

Schliep et al. 2010 developed a three-stage Bayesian hierarchical model for spatial modeling of areal data from regional climate models. We refer the reader to the original paper for a more complete description. The three stages of the model, namely the data, process and parameter stages, will be briefly described as follows.

### 2.1 Stage 1: Data model

We model sets of areal data $\{z_{it}\}$, a maxima at grid cell $i$ and time $t$ (in year), to follow a GEV distribution

$$P(Z_{it} \leq z) = \exp\left\{-\left(1+\xi_i\frac{z-\mu_i}{\sigma_i}\right)^{-1/\xi_i}\right\}$$

provided $(1+\xi_i\frac{z-\mu_i}{\sigma_i})>0$ where $\mu_i\in\mathbb{R}$, $\sigma_i>0$, and $\xi_i\in\mathbb{R}$ are unknown location, scale and shape parameters at site $i$. Let $\mathbf{z}=(\mathbf{z}_1^T,\mathbf{z}_2^T,\ldots,\mathbf{z}_n^T)^T$ where $\mathbf{z}_i^T$ is a vector of $t$-year independent maxima at grid cell $i$. Let $\boldsymbol{\mu}=(\mu_1,\ldots,\mu_n)$, and define analogously for $\boldsymbol{\sigma}$ and $\boldsymbol{\xi}$. Furthermore, assume $z_{it}|\mu_i,\sigma_i,\xi_i$ is conditionally independent of $z_{jt}|\mu_j,\sigma_j,\xi_j$ for $i\neq j$. Then the likelihood of the data, the first stage of the three-stage model is given as follows

$$L(\boldsymbol{\mu},\boldsymbol{\sigma},\boldsymbol{\xi}|\mathbf{z}) = \prod_i\prod_t \sigma_i^{-1}\left(1+\xi_i\frac{z_{it}-\mu_i}{\sigma_i}\right)^{-1/\xi_i-1}\exp\left\{-\left(1+\xi_i\frac{z_{it}-\mu_i}{\sigma_i}\right)^{-1/\xi_i}\right\}$$

The second stage is the process level in which each of the GEV parameters is assumed to follow normal distribution. The means comprise fixed and random effects, and the variances are assumed constant with the same values across cells. Consequently, we can write

$$\boldsymbol{\mu}\sim \mathrm{N}\left(\mathbf{X}_\mu^T\boldsymbol{\beta}_\mu+\boldsymbol{U}_\mu,\ \tau_\mu^{-2}\mathbf{I}\right),\quad \log(\boldsymbol{\sigma})\sim \mathrm{N}\left(\mathbf{X}_\sigma^T\boldsymbol{\beta}_\sigma+\boldsymbol{U}_\sigma,\ \tau_\sigma^{-2}\mathbf{I}\right),\quad \boldsymbol{\xi}\sim \mathrm{N}\left(\mathbf{X}_\xi^T\boldsymbol{\beta}_\xi+\boldsymbol{U}_\xi,\ \tau_\xi^{-2}\mathbf{I}\right)$$

where $\mathbf{X}_\theta$ is the design matrix; $\boldsymbol{\beta}_\theta$ is a vector of unknown regression coefficients, $\boldsymbol{U}_\theta$ is a spatial random effect, $\tau_\theta$ is fixed precision that has the same value for all sites, and $\theta$ is generically used to stand for $\mu,\sigma,$ and $\xi$. Here, the 4-vector of covariates $\boldsymbol{X}_{\theta,i}$ consist of rescaled latitude, rescaled longitude and rescaled elevation at site $i$.

It is natural that temperature extremes at one location would have a strong correlation with extremes at neighboring sites. The dependence among neighbors may be modeled by a conditional autoregressive (CAR) model, in which the data are assumed to follow some function depending on data in the neighborhood. Following Schliep et al. 2010 we also use a CAR to

model the spatial random effect in the hierarchical model. The relationship between the three GEV parameters $(\boldsymbol{\mu}, \boldsymbol{\sigma}, \boldsymbol{\xi})$ is modeled spatially via a random effect $\boldsymbol{U}_\theta$ using an intrinsic autoregressive (IAR). The IAR model is a special case of the CAR model. We used a proximity matrix of order one, $\mathbf{W} = \{w_{ij}\}$ in that $w_{ij} = 1$ if site $i$ and $j$ share a boundary, and $w_{ij} = 0$ otherwise. With this setup, a joint probability distribution of $\boldsymbol{U} = (\boldsymbol{U}_\mu^T, \boldsymbol{U}_\sigma^T, \boldsymbol{U}_\xi^T)^T$ would be normal, $\boldsymbol{U} \sim \mathrm{N}(\mathbf{0}, \mathbf{Q}^{-1})$, where $\mathbf{Q} = \mathbf{T}(\mathbf{D}_w - \mathbf{W})$, in which $\mathbf{D}_w$ is a diagonal matrix with $(d_w)_{ij} = w_{i+}$ and $\mathbf{T}$ is the covariance matrix of random effect $\boldsymbol{U}$.

### 2.2 Stage 3: Parameter model

In this stage, there are only two parameters left, $\boldsymbol{\beta} = \{\boldsymbol{\beta}_\mu, \boldsymbol{\beta}_\sigma, \boldsymbol{\beta}_\xi\}$ and $\mathbf{T}$. As three covariates (latitudes, longitudes and elevations) are included, $\boldsymbol{\beta}_\mu$ consists of four elements i.e. $\boldsymbol{\beta}_\mu = (\beta_{0\mu}, \beta_{1\mu}, \beta_{2\mu}, \beta_{3\mu})^T$. $\boldsymbol{\beta}_\sigma$ and $\boldsymbol{\beta}_\xi$ are similarly defined. In what follows we assume that the $\boldsymbol{\beta}$ have a Gaussian distribution. The prior for $\beta_{0\theta}$, where again $\theta$ represents $(\mu, \sigma, \xi)$, is assumed to follow an independent normal distribution with mean equal to the mean of cell-wise maximum likelihood estimate of the associated parameter with precision $100$. The priors for both $\beta_{1\theta}, \beta_{2\theta}$ and $\beta_{3\theta}$ are zero-mean normal distribution with precision $10$ respectively. These are conjugate priors, with conjugacy chosen to ease the computation. A Wishart distribution is known to be a conjugate prior of the inverse covariance matrix of a multivariate normal distribution, and hence a Wishart prior with 3 degrees of freedom is assigned to the precision matrix $\mathbf{T}$.

## 3 Implementation

We applied the model to the temperature maxima retrieved from the Tasmanian Partnership of Advanced Computing portal (http://dl.tpac.org.au/tpacportal/). The data are monthly temperature maxima ($^0$K) from1961 to 2009 and are generated by regional climate model of CSIRO (**?**) with high resolution of 0.1 degree and with A2 emissions scenario. The spatial domain consists of a $56 \times 51$ grid, $2856$ grid cells, covering Tasmania (Latitude: $44 - 39^0$S, Longitude: $143.5 - 149^0$E). We look for the future impact of the climate change on the regional scale. We used annual block maxima.

From the mean data plot (not shown here) we noted that the data exhibit spatial correlation. We assumed constant variance for each parameter, specified in terms of precision $(\tau_\mu^2, \tau_\sigma^2, \tau_\xi^2) = (0.1, 8, 350)$. These values reflect the scale differences of the GEV parameters, but they were also chosen so that most of the variability would have to be explained by the spatial random effect $\boldsymbol{U}$.

We used a hybrid MCMC that combines Metropolis-Hastings algorithm and Gibbs sampler to sample posterior parameters of interest. The former approach is used to sample the GEV parameters, while the latter is used to sample $\boldsymbol{\beta}, \boldsymbol{U}$ and $\mathbf{T}$. The GEV parameters in each cell are updated using a Metropolis-Hastings algorithm with target distribution:

$$p(\boldsymbol{\mu}, \boldsymbol{\sigma}, \boldsymbol{\xi} | \mathbf{z}, \boldsymbol{\mu}_0, \boldsymbol{\sigma}_0, \boldsymbol{\xi}_0) = \mathrm{GEV}(\mathbf{z} | \boldsymbol{\mu}, \boldsymbol{\sigma}, \boldsymbol{\xi}) \mathrm{N}(\boldsymbol{\mu} | \boldsymbol{\mu}_0) \mathrm{N}(\log(\boldsymbol{\sigma}) | \boldsymbol{\sigma}_0) \mathrm{N}(\boldsymbol{\xi} | \boldsymbol{\xi}_0).$$

As the second level and the third level of the model are assumed to follow a Normal distribution, we can derive the complete conditional distribution of each parameter analytically and use a

Gibbs sampler to generate samples from these distributions.

## 4 Graphical Results

We carried out 50000 iterations, the first 20000 were discarded as a burn-in period, and we took every $20$th observation to eliminate autocorrelation. The running mean plots for each chain of simulated parameter were well behaved, and converged to each specific value (graphs are not shown here). The trace plots of each simulated parameter again exhibit stationary with empirical density resembling the assumed distribution (normal).

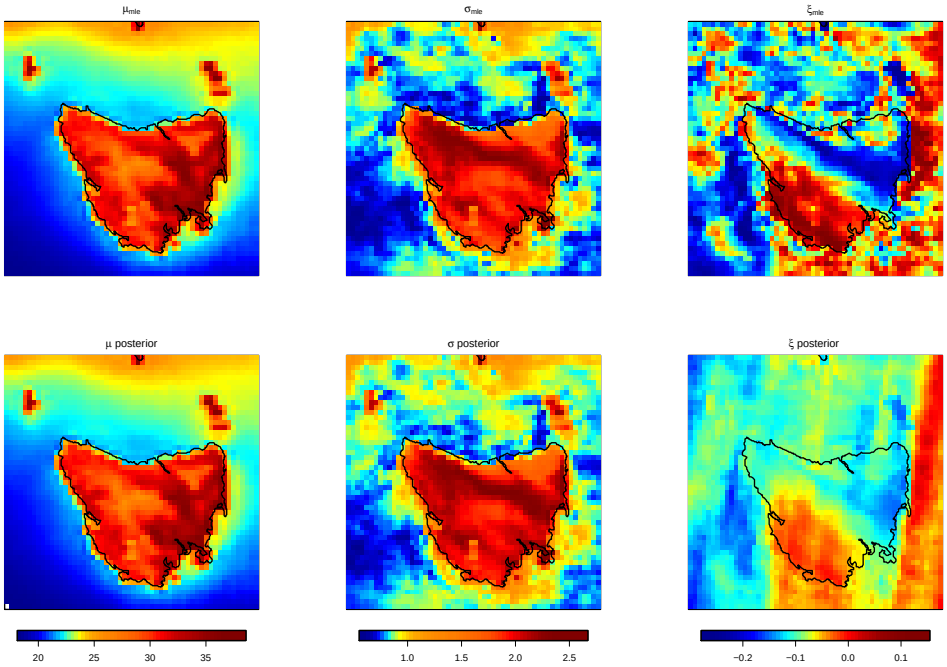


Figure 1: Cell-wise MLE estimates of GEV parameters (first row) and the corresponding posterior means calculated from the MCMC chains (second row).

Fig.1 shows cell-wise MLE estimates of GEV parameters (first row) and the corresponding posterior means calculated from the MCMC chains (second row). We hardly see the difference between cell-wise MLE parameter estimates and the corresponding posteriors for location and scale parameters. This presumably due to the fine scale grids ($0.1$ degree) type of the data, so the data themselves exhibit considerable smoothness. However, the spatial hierarchical model has successfully smoothed the shape parameters. As we can see from Fig.1, the estimates of shape parameter from the Bayesian model occupied a narrower range than those from cell-wise estimates.

Furthermore, as return level is of much interest to practitioners, we calculated the return level from both the spatial model and independent model. For the independent model, the $p$-year return level is calculated from the cell-wise MLE GEV estimates; it is simply the $1/p$ quantile of the fitted GEV distribution. For the spatial model, we sampled 5000 posterior parameters from the MCMC chains, from which we calculated $p$-year return levels. The resulting collections of return levels are used to compute credible intervals. We took the 0.025 and 0.975 quantiles

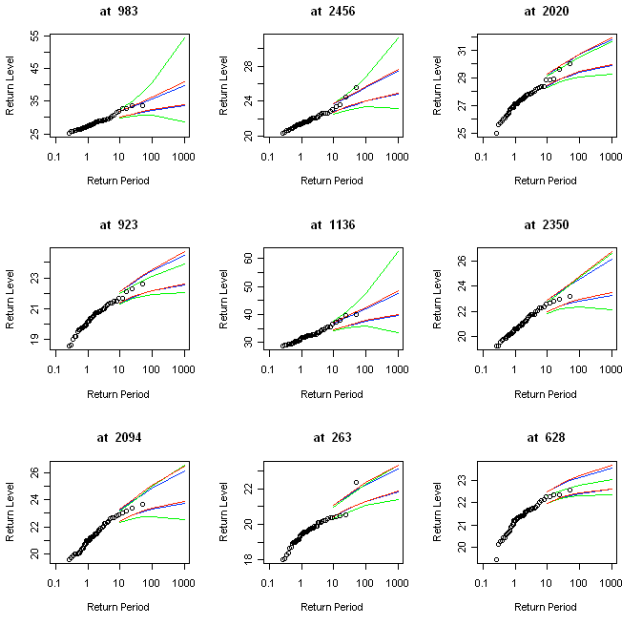


Figure 2: Credible Interval of p-year return levels calculated from the independent GEV model (green line) and from the spatial model. The red line is the 0.025 and 0.975 quantiles of the return levels calculated from spatial model while the blue line is the corresponding HPD

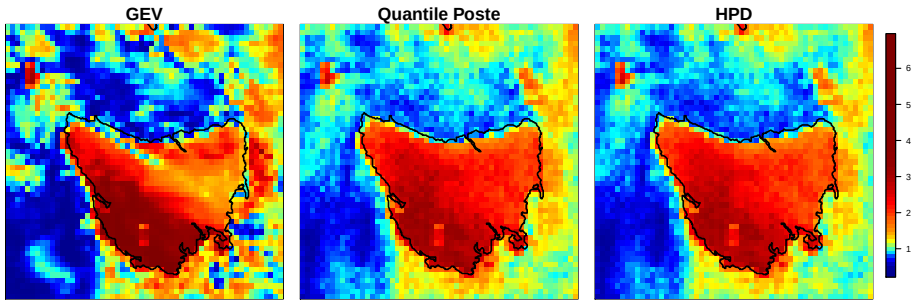


Figure 3: The width of 25-year return levels credible interval at each grid, calculated from cell-wise GEV parameter estimates (left), .25 and .75 quantile posterior parameters (middle), HPD of posterior parameters (right).

as for the lower and upper bound of credible interval of these return levels. In addition, we also computed the high probability density (HPD) from the collection of return levels. The three kinds of intervals at six locations are plotted in the same panel (Fig.2). From Fig.2 we can see, both the credible intervals and the HPD intervals calculated from the spatial model perform similarly, and at most locations the two intervals are narrower than that of independent model. In fact this is happened at most grid cells. We calculated 25-year return levels at each grid using independent estimates and spatial model estimates, calculated the corresponding credible interval and the HPD, then plotted widths of these intervals (Fig. 3). We can see from Fig.3 that the credible intervals from the spatial model are narrower than that of independent intervals at most cells. Thus, having the model has successfully smoothed the shape parameter; the spatial model produces more reasonable return levels than that of independent level.

## 5 Posterior Predictive Distribution

Simulation is one of the diagnostic tools for Bayesian model that is reasonable and convenient to apply. Here, we follow Gelman, Carlin, Stern and Rubin 1995 to conduct model checking through simulation of predictive posterior distribution and then comparing the statistics of interest from the posterior predictive distribution to the corresponding statistics of the observed data. Let denote $\tilde{z}$ a future observation. The posterior predictive distribution, $p(\tilde{z}|\mathbf{z})$ is found by integration $p(\tilde{z}|\mathbf{z}) = \int p(\tilde{z}|\boldsymbol{\theta})p(\boldsymbol{\theta}|\mathbf{z})d\boldsymbol{\theta}$. Gelman et al. 1995 defined two different types of posterior predictive distribution, namely 1) the distribution of future observations corresponding to the existing $\boldsymbol{\theta}$, and 2) the distribution of future observations corresponding to the future $\boldsymbol{\theta}$ drawn from the population parameter distribution. In case one, the future observation $\tilde{z}$ is simulated based on the existing $\boldsymbol{\theta}$, that is, the simulated chain of the MCMC output. In the latter case, the future parameter $\tilde{\boldsymbol{\theta}}$ is drawn based on the existing hyperparameters $\phi$, then the future observations are drawn based on $\tilde{\boldsymbol{\theta}}$. In this paper we used the second type of posterior predictive distribution. First, we drew $\tilde{\boldsymbol{\theta}}$ from the existing posterior hyperparameters $\phi$ from the conditional distribution $\tilde{\theta}|\boldsymbol{\beta}, \boldsymbol{U} \sim \mathrm{N}(\mathbf{X}^T\boldsymbol{\beta} + \boldsymbol{U},\ \tau^{-2}\mathbf{I})$, then, generated the future observations $\tilde{z}$ from $\tilde{z}|\tilde{\boldsymbol{\theta}} \sim \mathrm{GEV}(\tilde{\boldsymbol{\theta}})$. The statistics of interest were calculated from the generated future observations and were compared to the corresponding statistics of the observed data. We took the largest value over 49 future observations and compared it to the largest observed data for every cell. Fig.4 shows the histogram of the results from 1000 repeated processes at some sites. We can see that the observed value lies well in the distribution of the generated data, suggesting a good model.

## 6 Summary

We modeled temperature extremes using the three-stage hierarchical model of Schliep et al. 2010. We found that the spatial random effect has successfully smoothed the shape parameters, but this was not apparent for the location and scale parameters. The smooth shape parameters resulted in more reasonable credible intervals of return levels than that of independent model (cell-wise MLE estimates).

In general, the outcomes from the complex spatial hierarchical model were quite similar to the simple inexpensive cell-wise MLE estimates. This can be seen as good performance of the spatial model in the sense that the spatial model is a ”right” model. One possible reason of the close results of the two models is that large number of data available at each grid. MLE can perform very well with such large data, while for small data Bayesian is expected to outperform MLE.

The performance of the Monte Carlo method can be improved by choosing the best proposal distribution in Metropolis algorithm. From our experience in conducting MCMC simulation, the determination of $\tau^2$, a fixed precision for GEV parameters, considerably affects the rate of convergence. Other variables that greatly affect the MCMC chains are the choice of proposal distribution and the jump; too small or too big jump results in slower convergence and higher autocorrelation.

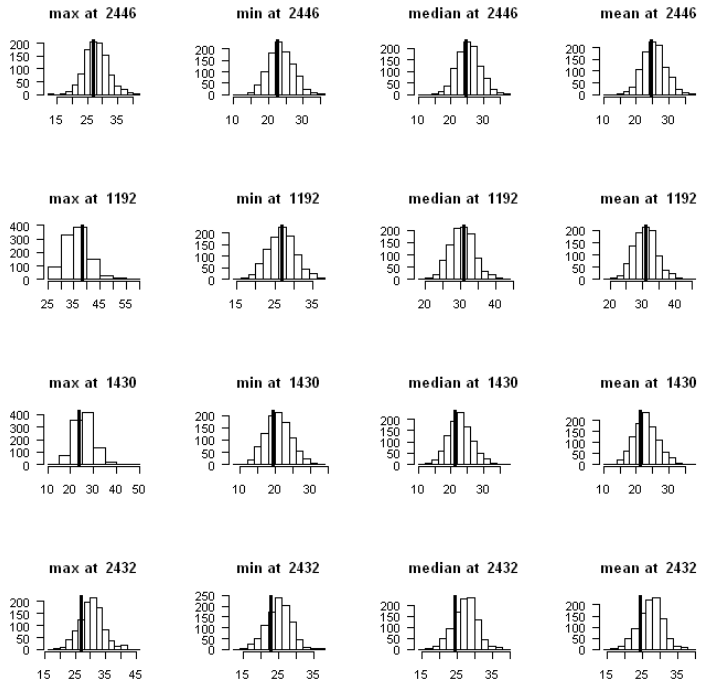


Figure 4: Histogram of 1000 replications of the maximum, minimum, median and mean of the 49 values generated from the posterior predictive distribution at several random locations. The vertical line indicates the corresponding statistics of the data

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## References

Banerjee, S., Carlin, B. and Gelfand, A. 2004. Hierarchical modeling and analysis for spatial data, *Monographs on statistics and applied probability (101)* .

Cooley, D., Nychka, D. and Naveau, P. 2007. Bayesian spatial modeling of extreme precipitation return levels, *Journal of the American Statistical Association* **102**(479): 824–840.

Gelman, A., Carlin, J. B., Stern, H. S. and Rubin, D. B. 1995. *Bayesian Data Analysis*, Chapman & Hall.

Hrafnkelsson, B., Morris, J. S. and Baladandayuthapani, V. 2012. Spatial modeling of annual minimum and maximum temperatures in iceland, *Meteorology and Atmospheric Physics* **116**(1-2): 43–61.

Kang, E. and Cressie, N. 2011. Bayesian inference for the spatial random effects model, *Journal of the American Statistical Association* **106**(495): 972–983.

Sang, H. and Gelfand, A. 2009. Hierarchical modeling for extreme values observed over space and time, *Environmental and Ecological Statistics* **16**(3): 407–426.

Schliep, E., Cooley, D., Sain, S. and Hoeting, J. 2010. A comparison study of extreme precipitation from six different regional climate models via spatial hierarchical modeling, *Extremes* **13**(2): 219–239.