Optimal deteriorating items production inventory models with random breakdown machine and stochastic repair time

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Abstract

This study develops deteriorating items production inventory models with random machine breakdown and stochastic repair time. The models assume that machine repair time is independent of machine breakdown rate. The classical optimization technique is used to derive an optimal solution. A numerical example and sensitivity analysis are shown to illustrate the models. The stochastic repair models with uniformly distributed repair time tends to have a larger optimal total cost than the fixed repair time model, however the production up time is less than the fixed repair time model. Production and demand rate are the most sensitive parameters for the optimal production up time, and demand rate is the most sensitive parameter for the optimal total cost of the stochastic model with exponential distribution repair time.

Keywords: inventory, machine breakdown, deteriorating items, stochastic repair time.

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1. Introduction

The production inventory problem has been investigated in recent years, and increasing number of researchers analyzes machine breakdown effect in production inventory problem. The effects of machine breakdown and corrective maintenance on the economic lot sizing were studied by Groenevelt *et al.* [1]. Groenevelt *et al.* [2] did similar research and included safety stocks to meet the service level. Moinzadeh and Aggarwal [3] proposed an (s, S) model assuming the time between breakdowns is exponentially distributed, the restoration times are constant and excess demand is backordered. The research was extended by Arreola-Risa and DeCroix [4] who assumed shortages are partially backordered. Aboud [5] later extended the research by Groenevelt *et al.*[1]. In his model, he assumed machine failure may occur during production. When a machine fails, it will be repaired immediately. He assumed that repair times are independent and identically distributed. Giri *et al.* [6] developed EMQ model with machine failure and general repair time. They proposed a model to determine the production rate and production lot size to minimize the annual expected total cost.

Recently, many researchers extended the production inventory with machine breakdown model by considering other problems in production such as deterioration, preventive maintenance and rework. In the deteriorating production process, the process may move from an in-control state to an out-of-control state, and produces some proportion of defective items. Kim and Hong [7] developed Economic Manufacturing Quantity (EMQ) model with three deteriorating processes. Wang [8] developed an EPQ mathematical model where production shifts from an in-control state to an out-of-control state with a general shift distribution. Chakraborty *et al*. [9] developed an EPQ model considering production system that may shift from an in-control state to an out-of-control state or may breakdown at any random time during a production period.

Preventive maintenance is usually used to reduce machine breakdown. Incorporated preventive maintenance to production inventory model was done by Cheung and Hausman [10]. They developed a mathematical model with random machine breakdowns and considered preventive maintenance and safety stock. Later Dohi *et al.* [11] extended the research by extending the model by Cheung and Hausman [10] to consider the stochastic nature of the model. Giri and Dohi [12] developed EMQ model with random variables corrective and preventive repair. They proposed solution procedure and computational algorithms to find the optimal production rate and lot size. El-Ferik [13] developed an EPQ model for unreliable manufacturing facility. Similar research for EPQ model with imperfect process has been done by Liao *et al*. [14]. Chiu *et al*. [15] developed an EPQ model with scrap, rework and stochastic machine breakdowns.

Deterioration is defined as decay, evaporation, obsolescence, and loss of quality marginal value of commodity that result in decreasing usefulness from its original condition. Some items like vegetables, milks, and fruits have deteriorating characteristics. Many researchers focus their study on deteriorating item, such as Balkhi [16], Chang *et al.* [17] and Roy *et al.* [18]. Misra [19] developed an EPQ for deteriorating items model and Wee [20] extended the model by considering partial backorder. Liao [21] developed a production inventory model for deteriorating item taking into consideration the effect of permissible delay in payment. An EPQ model for deteriorating production equipment and items was developed by Alfares *et al*. [22]. In their model, they also considered quality, inspection and maintenance, varying demand and production rates. Lo *et al.* [23] developed integrated production and inventory model. The model assumed varying deterioration rate, partial backordering, inflation, imperfect production process and multiple deliveries.

From the authors' literature search, very few researchers have considered production inventory model with deteriorating item and stochastic machine breakdown. Lin and Gong [24] developed EPQ deteriorating inventory model with machine breakdown and fix repair time. In real life, most repair times are stochastic. This paper extends the excellent research initiated by Lin and Gong [24]. Here, two models are developed: one model assumes repair time is uniformly distributed and the other model assumes repair time is exponentially distributed. The paper has four sections. The first section discusses the research motivation and literature review. The second section is model development. Section three shows an example and sensitivity analysis, and the final section is the concluding remarks.

2. Model development

The assumptions:

1. Production rate is greater than demand rate.
2. Production and demand rate are constant.
3. Deteriorating rate is constant.
4. There is no repair or replacement for a deteriorated item.
5. Machine repair time is independent of machine breakdown.

Notations:

I = inventory level

I_1 = inventory level in production period

I_2 = inventory level in non-production period

T_1 = production period

T_2 = non production period

T_3 = shortage period

T_p = machine breakdown period

p = production rate

d = demand rate

θ = deteriorating rate

K = setup cost

h = holding cost

S = lost sales cost

π = deteriorating cost

M = repair machine cost

The inventory policy for lost sales case is illustrated in Figure 1. When the machine does not breakdown during the production period, the production is performed during T_1 time period. When inventory reaches maximum level I_m, the production stops and inventory decreases due to demand and deterioration. The inventory level reaches zero units at time (T_1+T_2), and machine starts to produce the item again. Since the machine has a possibility of breakdown, the machine may run the whole T_1 period. When breakdown occurs, the production period time is T_p time. When a machine breaks down, it will require some repair time. Since repair time is stochastic in (T_2+T_3) period, production may not always be possible and lost sales may occur during T_3 time period.

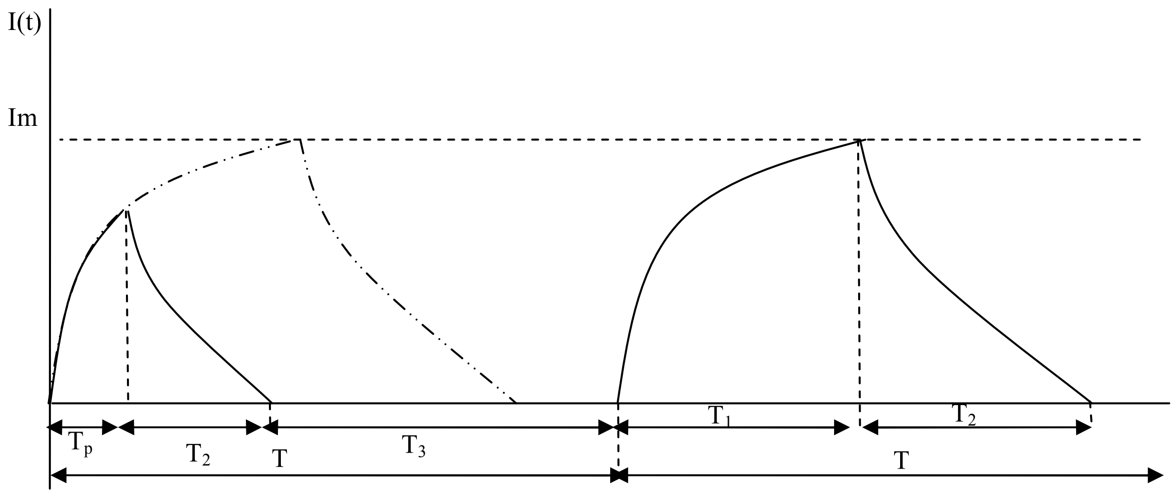

Figure 1. Inventory level of lost sales case

The number of inventory in production period from the problem above can be formulated as:

$$\Delta I_1(t) + \theta I_1(t_1) = p - d \qquad 0 \le t_1 \le T_1 \tag{1}$$

While the number of inventory level in non production period is represented by the following equation:

$$\Delta I_2(t) + \theta I_2(t_2) = -d \qquad 0 \le t_2 \le T_2 \tag{2}$$

Since $I_1(0)$ = 0, and $I_2(T_2)$ = 0, the inventory level in production period and non production period are:

$$I_1(t_1) = \frac{p-d}{\theta}(1 - e^{-\theta t_1}) \qquad 0 \le t_1 \le T_1 \tag{3}$$

$$I_2(t_2) = \frac{d}{\theta}(e^{\theta(T_2 - t_2)} - 1) \qquad 0 \le t_2 \le T_2 \tag{4}$$

Since $I_1 = I_2$ when $t_1 = T_1$ and $t_2 = 0$ then one has:

$$\frac{p-d}{\theta}(1 - e^{-\theta T_1}) = \frac{d}{\theta}(e^{\theta T_2} - 1) \tag{5}$$

Using the Taylor series approximation (Yang and Wee [25]), one has:

$$(p-d)(T_1 - \frac{1}{2}\theta T_1^{\,2}) = d(T_2 + \frac{1}{2}\theta T_2^{\,2}) \tag{6}$$

Since θ T_2 is a small number, T_2 in terms of T_1 can be approximated as:

$$T_2 \cong \frac{(p-d)T_1(1-\frac{1}{2}\theta T_1)}{d} \tag{7}$$

This approximation is commonly used in deteriorating production inventory problem (see Misra [19]). The expected inventory level can be expressed by the following:

$$E(I) = \int_{t=0}^{T_2} \frac{p-d}{\theta}(1-e^{-\theta t_1}) + \int_{t=0}^{T_2} \frac{d}{\theta}(e^{\theta(T_2-t_2)}-1)$$

$$E(I) = \frac{p-d}{\theta^2}\left(\theta T_1 + e^{-\theta T_1} - 1\right) + \frac{d}{\theta^2}\left(-\theta T_2 + e^{\theta T_2} - 1\right) \tag{8}$$

Using Taylor series approximation, (8) can be written as:

$$E(I) = (p-d)\left(\frac{T_1^{\,2}}{2}\right) + d\left(\frac{T_2^{\,2}}{2}\right) \tag{9}$$

Substitute (7) to (9), one has:

$$E(I) = p-d\left(\frac{T_1^{\,2}}{2}\right) + d\left(\frac{(p-d)T_1(1-\frac{1}{2}\theta T_1)}{2d}\right)^2 \tag{10}$$

Since $\frac{1}{2}\theta T_1$ is very small, (10) can be simplified as:

$$E(I) = \frac{p^2}{2d}\left(1-\frac{d}{p}\right)T_1^{\,2} \tag{11}$$

Since there is breakdown machine possibilities, then (11) can be formulated as:

$$E(I) \cong \begin{cases} \frac{p^2}{2d}(1-\frac{d}{p})T_p^2 & for \;\; T_p \le T_1 \\ \frac{p^2}{2d}(1-\frac{d}{p})T_1^2 & for \;\; T_p > T_1 \end{cases} \tag{12}$$

Here, similar breakdown machine probability density function as Lin and Gong [24] is used. The function of T_p is equal to $f(T_p) = \mu e^{-\mu T_p}, \; for \; T_p > 0$. The expected inventory can be written as:

$$E(I) = \int_{T_p=0}^{T_1} \frac{p^2}{2d}(1-\frac{d}{p})T_p^2 \mu e^{-\mu T_p} + \int_{T_p=T_1}^{\infty} \frac{p^2}{2d}(1-\frac{d}{p})T_1^2 \mu e^{-\mu T_p}$$

$$E(I) = \frac{p(p-d)\left(1-e^{-\mu T_1} - \mu T_1 e^{-\mu T_1}\right)}{\mu^2 d} \tag{13}$$

The expected deteriorated item can be formulated as:

$$E(R) = pT_1 - d(T_1 + T_2) \tag{14}$$

Substitute T_2 from (7) to (14), then it can be simplified as:

$$E(R) \cong \begin{cases} \frac{p}{2}(1-\frac{d}{p})\theta T_p^2 & for \; T_p \le T_1 \\ \frac{p}{2}(1-\frac{d}{p})\theta T_1^2 & for \; T_p > T_1 \end{cases} \tag{15}$$

Using the similar function of T_p, the expected deteriorating item can be written as:

$$E(R) = \int_{T_p=0}^{T_1} \frac{p}{2}(1-\frac{d}{p})\theta T_p^2 \mu e^{-\mu T_p} + \int_{T_p=T_1}^{\infty} \frac{p}{2}(1-\frac{d}{p})\theta T_1^2 \mu e^{-\mu T_p}$$

$$E(R) = \frac{(p-d)\theta\left(1-e^{-\mu T_1} - \mu T_1 e^{-\mu T_1}\right)}{\mu^2} \tag{16}$$

The corrective cost can be formulated as:

$$E(Mc) = M(1-e^{-\mu T_1}) \tag{17}$$

The total inventory cost of lost sales case consists of setup cost, corrective cost, holding cost, deteriorating cost and lost sales cost. Lost sales occur when machine repair time period is longer than non production time period. The total inventory cost can be expressed as:

$$TC(T_1,T_2) = E\left[K + M(1-e^{-\mu T_1}) + hE(I) + \pi E(R) + Sd\int_{T_p=0}^{T_1}\int_{t=T_2}^{\infty}(t-T_2)f(t)\mu e^{-\mu T_p}\,dtdT_p\right]$$

$$TC(T_1,T_2) = E\left[K + M(1-e^{-\mu T_1}) + (h\frac{p}{d} + \pi\theta)\left(\frac{(p-d)\left(1-e^{-\mu T_1} - \mu T_1 e^{-\mu T_1}\right)}{\mu^2}\right) + Sd\int_{T_p=0}^{T_1}\int_{t=T_2}^{\infty}(t-T_2)f(t)\mu e^{-\mu T_p}\,dtdT_p\right]$$

(18)

The total replenishment time is equal to the production up time period, non production period and repair time probability. The expected total replenishment time can be formulated as:

$$E(T) = \int_{T_p=0}^{T_1} (T_1 + T_2 + \int_{t=T_2}^{\infty} (t - T_2) f(t) dt) \mu e^{-\mu T_p} dT_p \tag{19}$$

The expected T_1 and T_2 time period can be written as:

$$E(T_1 + T_2) = \int_{T_p=0}^{T_1} \left(1 + (p-d) \frac{(1 - 0.5\theta T_p)}{d} \right) T_p \mu e^{-\mu T_p} dT_p + \int_{T_p=T_1}^{\infty} \left(1 + (p-d) \frac{(1 - 0.5\theta T_1)}{d} \right) T_1 \mu e^{-\mu T_p} dT_p$$

$$E(T_1 + T_2) = \frac{p(1 - e^{-\mu T_1})}{d\mu} \tag{20}$$

Substitute (20) to (19), one has:

$$E(T) = \frac{p(1 - e^{-\mu T_1})}{d\mu} + \int_{T_p=0}^{T_1} \int_{t=T_2}^{\infty} (t - T_2) f(t) \mu e^{-\mu T_p} dt dT_p \tag{21}$$

Using the renewal reward theorem, the expected total cost per unit time can be formulated as follows:

$$TCT = \frac{E(TC)}{E(T)} \tag{22}$$

Combine (18) and (21), we have the expected total cost per unit time as follows:

$$TCT(T_1, T_2) = \frac{K + M(1 - e^{-\mu T_1}) + (h\frac{p}{d} + \pi\theta)\left(\frac{(p-d)\left(1 - e^{-\mu T_1} - \mu T_1 e^{-\mu T_1}\right)}{\mu^2} \right) + Sd \int_{T_p=0}^{T_1} \int_{t=T_2}^{\infty} (t - T_2) f(t) \mu e^{-\mu T_p} dt dT_p}{\frac{p(1 - e^{-\mu T_1})}{d\mu} + \int_{T_p=0}^{T_1} \int_{t=T_2}^{\infty} (t - T_2) f(t) \mu e^{-\mu T_p} dt dT_p} \tag{23}$$

2.1. Uniform distribution case

Assume that the machine repair time t, is a random variable that is uniformly distributed over the interval [0, b]. The probability density function, $f(t)$, is given as:

$$f(t) = \begin{cases} 1/b, & 0 \le t \le b \\ 0, & otherwise \end{cases} \tag{24}$$

Substitute uniform probability density function in (24) to (23), one has:

$$TCT(T_1,T_2)=\frac{K+M(1-e^{-\mu T_1})+(h\frac{p}{d}+\pi\theta)\left(\frac{(p-d)\left(1-e^{-\mu T_1}-\mu T_1e^{-\mu T_1}\right)}{\mu^2}\right)+\frac{Sd}{b}\int\limits_{T_p=0}^{T_1}\int\limits_{t=T_2}^{b}(t-T_2)\mu e^{-\mu T_p}\,dtdT_p}{\frac{p(1-e^{-\mu T_1})}{d\mu}+\frac{1}{b}\int\limits_{T_p=0}^{T_1}\int\limits_{t=T_2}^{b}(t-T_2)\mu e^{-\mu T_p}\,dtdT_p} \tag{25}$$

The expected shortage can be written as:

$$E(T_3)=\frac{1}{b}\int\limits_{T_p=0}^{T_1}\int\limits_{t=T_2}^{b}(t-T_2)\mu e^{-\mu T_p}\,dtdT_p$$

$$E(T_3)=\int\limits_{T_p=0}^{T_1}\frac{\left(b-\frac{(p-d)T_p(1-\frac{1}{2}\theta T_p)}{d}\right)^2}{2b}\mu e^{-\mu T_p}\,dTp \tag{26}$$

Simplify equation above, one has:

$$E(T_3)=\int\limits_{T_p=0}^{T_1}\frac{\left(b-\frac{(p-d)T_p}{d}\right)^2}{2b}\mu e^{-\mu T_p}\,dTp$$

$$E(T_3)=\frac{\left((b\mu d)^2-2b(p-d)\mu d+2(p-d)^2\right)\left(1-e^{-\mu T_1}\right)-e^{-\mu T_1}\mu T_1(p-d)\left((p-d)(2+\mu T_1)-2b\mu d\right)}{2b\mu^2d^2} \tag{27}$$

Substitute (27) to (25), the total cost per unit time of uniform distribution repair time is:

$$TCT=\frac{K+M(1-e^{-\mu T_1})+h\frac{p(p-d)\left(1-e^{-\mu T_1}-\mu T_1e^{-\mu T_1}\right)}{\mu^2d}+\pi\theta\frac{(p-d)\theta\left(1-e^{-\mu T_1}-\mu T_1e^{-\mu T_1}\right)}{\mu^2}}{\frac{p(1-e^{-\mu T_1})}{d\mu}+\frac{\left((b\mu d)^2-2b(p-d)\mu d+2(p-d)^2\right)\left(1-e^{-\mu T_1}\right)-e^{-\mu T_1}\mu T_1(p-d)\left((p-d)(2+\mu T_1)-2b\mu d\right)}{2b\mu^2d^2}}$$
$$+\frac{Sd\left(\frac{\left((b\mu d)^2-2b(p-d)\mu d+(p-d)^2\right)\left(1-e^{-\mu T_1}\right)-e^{-\mu T_1}\mu T_1(p-d)\left((p-d)(2+\mu T_1)-2b\mu d\right)}{2b\mu^2d^2}\right)}{\frac{p(1-e^{-\mu T_1})}{d\mu}+\frac{\left((b\mu d)^2-2b(p-d)\mu d+2(p-d)^2\right)\left(1-e^{-\mu T_1}\right)-e^{-\mu T_1}\mu T_1(p-d)\left((p-d)(2+\mu T_1)-2b\mu d\right)}{2b\mu^2d^2}} \tag{28}$$

Differentiate (28) with respect to T_1, one has:

$$\frac{\partial TCT(T_1)}{\partial T_1}=\frac{M\mu e^{-\mu T_1}+\left(h\frac{p}{d}+\pi\theta\right)\left((p-d)T_1e^{-\mu T_1}\right)}{\frac{p(1-e^{-\mu T_1})}{d\mu}+\frac{\left((b\mu d)^2-2b(p-d)\mu d+2(p-d)^2\right)\left(1-e^{-\mu T_1}\right)-e^{-\mu T_1}\mu T_1(p-d)\left((p-d)(2+\mu T_1)-2b\mu d\right)}{2b\mu^2d^2}}$$

$$
-\frac{\dfrac{S\mu^3 e^{-\mu T_1}\left(bd\left(-bd+2(p-d)T_1\right)-T_1^2(p-d)^2\right)}{2db\mu^2}}{\dfrac{p(1-e^{-\mu T_1})}{d\mu}+\dfrac{\left((b\mu d)^2-2b(p-d)\mu d+2(p-d)^2\right)\left(1-e^{-\mu T_1}\right)-e^{-\mu T_1}\mu T_1(p-d)\left((p-d)(2+\mu T_1)-2b\mu d\right)}{2b\mu^2 d^2}}
$$

$$
-\frac{A\left(K+M(1-e^{-\mu T_1})-\dfrac{\left(h\dfrac{p}{d}+\pi\theta\right)\left((p-d)\left(\mu T_1 e^{-\mu T_1}+e^{-\mu T_1}-1\right)\right)}{\mu^2}\right)}{\left(\dfrac{p(1-e^{-\mu T_1})}{d\mu}+\dfrac{\left((b\mu d)^2-2b(p-d)\mu d+(p-d)^2\right)\left(1-e^{-\mu T_1}\right)-e^{-\mu T_1}\mu T_1(p-d)\left((p-d)(2+\mu T_1)-2b\mu d\right)}{2b\mu^2 d^2}\right)^2}
$$

$$
-\frac{A\left(Sd\left(\dfrac{\left((b\mu d)^2-2b(p-d)\mu d+(p-d)^2\right)\left(1-e^{-\mu T_1}\right)-e^{-\mu T_1}\mu T_1(p-d)\left((p-d)(2+\mu T_1)-2b\mu d\right)}{2b\mu^2 d^2}\right)\right)}{\left(\dfrac{p(1-e^{-\mu T_1})}{d\mu}+\dfrac{\left((b\mu d)^2-2b(p-d)\mu d+(p-d)^2\right)\left(1-e^{-\mu T_1}\right)-e^{-\mu T_1}\mu T_1(p-d)\left((p-d)(2+\mu T_1)-2b\mu d\right)}{2b\mu^2 d^2}\right)^2} \tag{29}
$$

where:

$$
A=\frac{\left(\mu^3 e^{-\mu T_1}\left(bd(-bd+T_1(p-d)(2-(p-d)))\right)\right)}{2b\mu^2 d^2}
$$

Property 1

For a function f: $S \rightarrow R_1$ defined by $f(x) = g(x)/h(x)$ where g: $S \rightarrow R_1$ and h: $S \rightarrow R_1$, and S is a nonempty convex set in E_n.

The followings discuss convexity and concavity functions:

(a) g is convex on S, and $g(x) \geq 0$ for each x $\in$ S.

(b) h is concave on S, and $h(x) \geq 0$ for each x $\in$ S, and

(c) Both g and h are differentiable.

Using property 1, the total cost per unit time is convex where $0 \leq T_2 \leq b$, and

$$
\mathrm{M} < \frac{(\mathrm{hp}+\pi \mathrm{d}\theta)(8(\mathrm{p-d})-4\mathrm{b}\mu \mathrm{d}(2+\mathrm{b}\theta))-\mathrm{S}\mu \mathrm{bd}\theta(4(\mathrm{p-d})+\mu \mathrm{b}^2\ \mathrm{d}\theta)}{\mu^2 d}
$$

Detailed calculation is shown in Appendix A.

2.2. Exponential distribution case

In the second case, the machine repair time is a random variable that is exponentially distributed. Exponential probability density function with mean $\dfrac{1}{\lambda}$, is given as:

$$
f(t)=\lambda e^{-\lambda t} \qquad for\ \ \lambda > 0
$$

The expected shortage period is:

$$E(T_3) = \int_{T_p=0}^{T_1} \int_{t=T_2}^{\infty} (t - T_2)\lambda e^{-\lambda t} \mu e^{-\mu T_p} \, dt dT_p$$

$$E(T_3) = \frac{e^{T_1\left(\frac{\lambda(p-d)}{2d}(-2+\theta T_1)+\frac{\lambda p}{d}+\mu\right)} - e^{\lambda T_1\left(\frac{T_1 p\theta}{2d}+1-\frac{\theta T_1}{2}\right)}}{e^{T_1\left(\frac{\lambda p}{d}+\mu\right)}} \tag{30}$$

From (30), the value of the exponential distribution expected total time can be expressed as:

$$E(T) = E(T_1 + T_2) + E(T_3)$$

$$E(T) = \frac{p(1-e^{-\mu T_1})}{d\mu} + \frac{e^{T_1\left(\frac{\lambda(p-d)}{2d}(-2+\theta T_1)+\frac{\lambda p}{d}+\mu\right)} - e^{\lambda T_1\left(\frac{T_1 p\theta}{2d}+1-\frac{\theta T_1}{2}\right)}}{e^{T_1\left(\frac{\lambda p}{d}+\mu\right)}} \tag{31}$$

The total cost for exponential distribution case can be formulated by substituting (31) into (23), and one has:

$$TCT(T_1) = \frac{K + M(1-e^{-\mu T_1}) + (h\frac{p}{d} + \pi\theta)\left(\frac{(p-d)\left(1-e^{-\mu T_1} - \mu T_1 e^{-\mu T_1}\right)}{\mu^2}\right)}{\frac{p(1-e^{-\mu T_1})}{d\mu} + \frac{e^{T_1\left(\frac{\lambda(p-d)}{2d}(-2+\theta T_1)+\frac{\lambda p}{d}+\mu\right)} - e^{\lambda T_1\left(\frac{T_1 p\theta}{2d}+1-\frac{\theta T_1}{2}\right)}}{e^{T_1\left(\frac{\lambda p}{d}+\mu\right)}}}$$

$$+ \frac{Sd\left(\frac{-e^{\lambda T_1(\frac{T_1 p\theta}{2d}+1-\frac{\theta T_1}{2})} + e^{T_1\left(\frac{\lambda(p-d)}{2d}(-2+\theta T_1)+\frac{\lambda p}{d}+\mu\right)}}{e^{T_1\left(\frac{\lambda p}{d}+\mu\right)}}\right)}{\frac{p(1-e^{-\mu T_1})}{d\mu} + \frac{e^{T_1\left(\frac{\lambda(p-d)}{2d}(-2+\theta T_1)+\frac{\lambda p}{d}+\mu\right)} - e^{\lambda T_1\left(\frac{T_1 p\theta}{2d}+1-\frac{\theta T_1}{2}\right)}}{e^{T_1\left(\frac{\lambda p}{d}+\mu\right)}}} \tag{32}$$

The derivative of (32) with respect to T_1 is:

$$\frac{\partial TCT(T_1)}{\partial T_1}=\frac{\left(M\mu e^{-\mu T_1}+\left(h\frac{p}{d}+\pi\theta\right)\left(h(p-d)T_1e^{-\mu T_1}\right)\right)\left(e^{T_1(\frac{\lambda p}{d}+\mu)}\right)-Sd\left(\lambda(\frac{T_1p\theta}{d}+1-\theta T_1+\frac{p}{d}+\frac{\mu}{\lambda})A_e+\left(\frac{\lambda(d-p)}{d}+\frac{(p-d)\lambda\theta T_1}{d}\right)B_e\right)}{\left(\frac{p\left(1-e^{(-\mu T_1)}\right)}{d\mu}\right)\left(e^{T_1(\frac{\lambda p}{d}+\mu)}\right)-A_e+B_e}$$

$$\frac{\left(\left(K+M(1-\mu e^{-\mu T_1})+\left(h\frac{p}{d}+\pi\theta\right)\left(\frac{(p-d)(1-e^{-\mu T_1}-\mu T_1e^{-\mu T_1})}{\mu^2}\right)\right)\left(e^{T_1(\frac{\lambda p}{d}+\mu)}\right)-\frac{Sde^{\frac{-\lambda(p-d)T_1}{d}}}{\lambda}\right)Ce}{\left(\left(\frac{p\left(1-e^{(-\mu T_1)}\right)}{d\mu}\right)\left(e^{T_1(\frac{\lambda p}{d}+\mu)}\right)-A_e+B_e\right)^2} \tag{33}$$

where:

$$Ae=e^{\lambda T_1\left(\frac{T_1\theta p}{d}+1-0.5\theta T_1\right)}$$

$$Be=e^{T_1\left(\frac{\lambda(d-p)}{d}+\frac{(p-d)\lambda\theta T_1}{2d}+\frac{\lambda p}{d}+\mu\right)}$$

$$Ce=\frac{pe^{-\mu T_1}}{d}-\frac{\lambda(\frac{T_1p\theta}{2d}+1-0.5\theta T_1-\frac{p}{d}-\frac{\mu}{\lambda})e^{Ae}+\left(\frac{\lambda(d-p)}{d}+\frac{(p-d)\lambda\theta T_1}{d}\right)e^{Be}}{\left(e^{T_1(\frac{\lambda p}{d}+\mu)}\right)}$$

The optimal T_1 can be derived by equating (33) to zero. The optimal T_1 can be solved using a simple line search method since the total cost function is convex for small value of λ, μ, and T_1 under these conditions:

$$\lambda<\frac{(\mathrm{d}\mu+\mathrm{p})}{2(\mathrm{p}-\mathrm{d})}$$

and

$$\mathrm{M}<\frac{\left(\frac{\mathrm{hp}}{\mathrm{d}}+\pi\theta)\right)(\mathrm{p}-\mathrm{d})-\mathrm{Sd}\left(2\left(\frac{\lambda\mu(\mathrm{p}-\mathrm{d})}{\mathrm{d}}\right)-\mu^2\right)}{\mu^2}$$

The detailed calculation is shown in appendix B.

3. Example and sensitivity analysis

Similar data as Lin and Gong [24] is used to illustrate the uniform distribution repair time model and to compare the model with the fixed repair time model. In the comparison, the uniform distribution repair time mean rate is equal to the fixed repair time. Considering

K= \$ 50 per production cycle, M = \$ 200 per repair, p = 10,000 units per unit time, d = 7,500 units per unit time, h = \$ 1 per unit per unit time, S = \$ 5 per unit, θ= 0.2, unavailability time is uniformly distributed over the interval [0, 0.1], μ = 0.2 and the fixed repair time equals to 0.05. Here, Maple 8 is used to solve (29) resulting in T_1 = 0.202. The optimal total cost per unit time can be found by substituting T_1 in (28) resulting in TCT = \$ 640.8. In the study, the performance of the stochastic repair time with the fixed repair time for different lost sales and repair times is compared. Figure 2 shows the optimal production period. The figure shows that the production time of the stochastic model is shorter than the fix model for small lost sales cost and longer for high lost sales cost. The optimal production time of the stochastic model is more sensitive to the lost sales costs. Figure 3 shows the total cost of the stochastic model and the fix model for different lost sales costs.

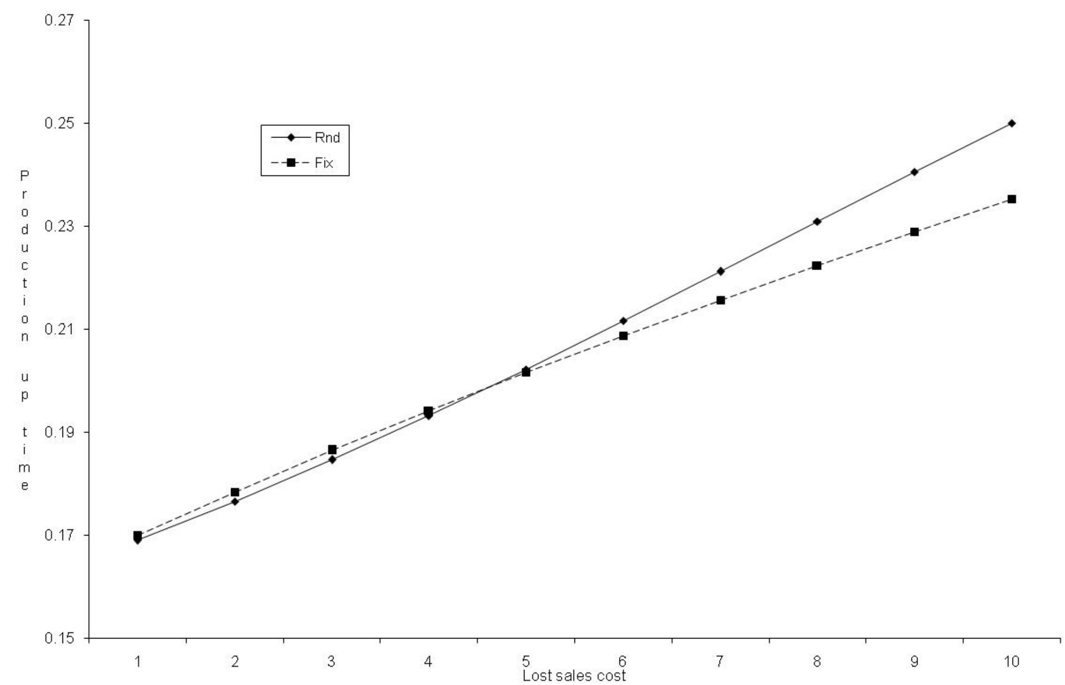

Figure 2. The comparison of T_1 in different lost sales cost

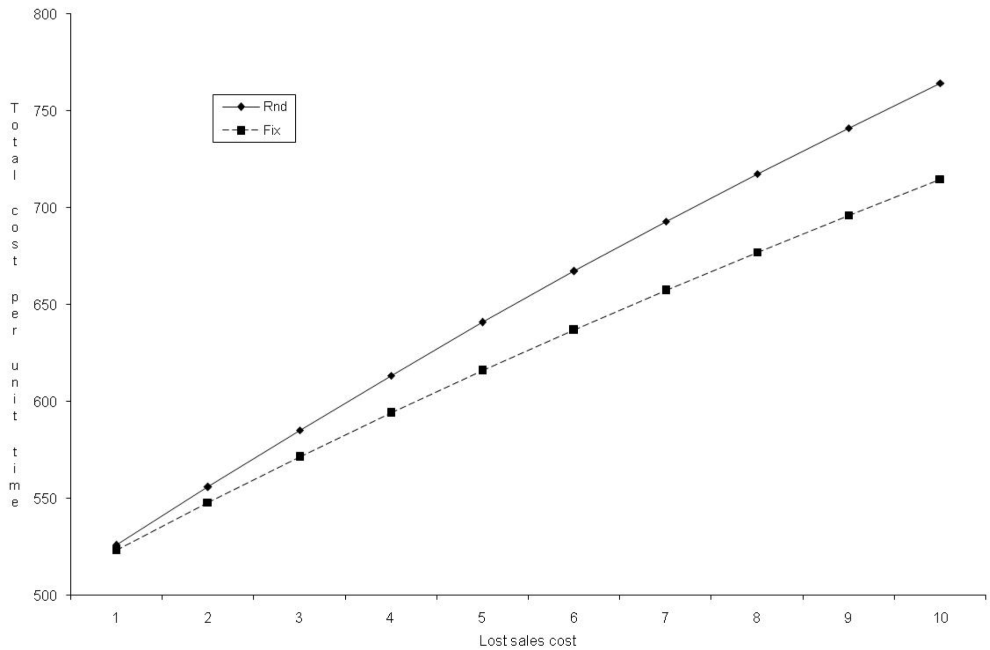

Figure 3. The comparison of the total cost in different lost sales

Figure 4 shows the production period of the stochastic model and the fix model for different repair times. The figure shows that the optimal production time of the stochastic model is less sensitive in varying repair time than the fix model. Also, the figure shows that the optimal production time difference between the stochastic model and the fix model is wider when the repair time is increased. Figure 5 shows the total cost of the stochastic model and the fix model for varying repair time.

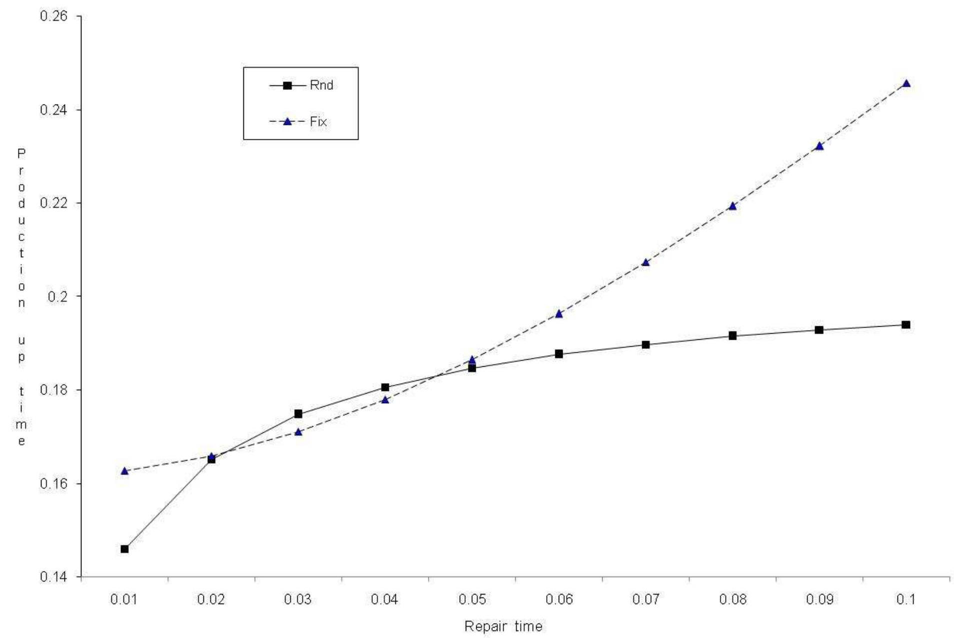

Figure 4. The comparison of T_1 for different repair time

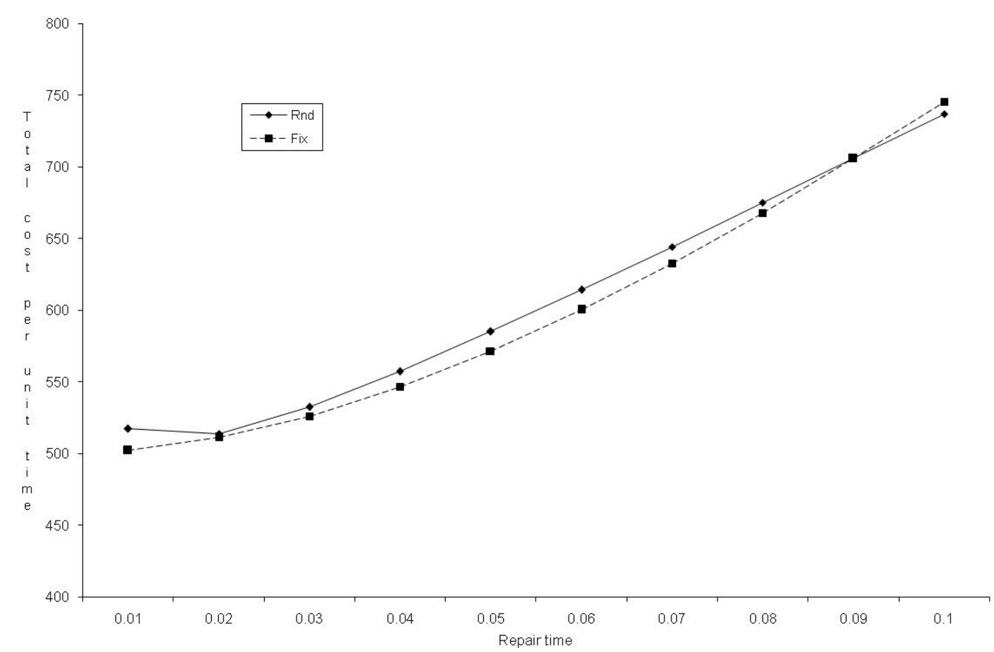

Figure 5. The comparison of the total cost in different repair time

The sensitivity analysis of the stochastic repair model with exponential distribution repair time uses similar data as the uniform distribution repair time model except that the maintenance rate (λ) equals to 20 is used. The sensitivity analysis is conducted by varying the data from -20% to 20%.

Figure 6 shows that the production rate is the most important parameter in decision making since it is the most sensitive parameter of the optimal production up time.

Figure 7 shows the sensitivity analysis of the total cost per unit time for various parameter values; the total cost is sensitive to the change in production rate, demand rate, holding cost and repair time rate. The total cost tends to increase as the parameter values increase except for repair time and production rate.

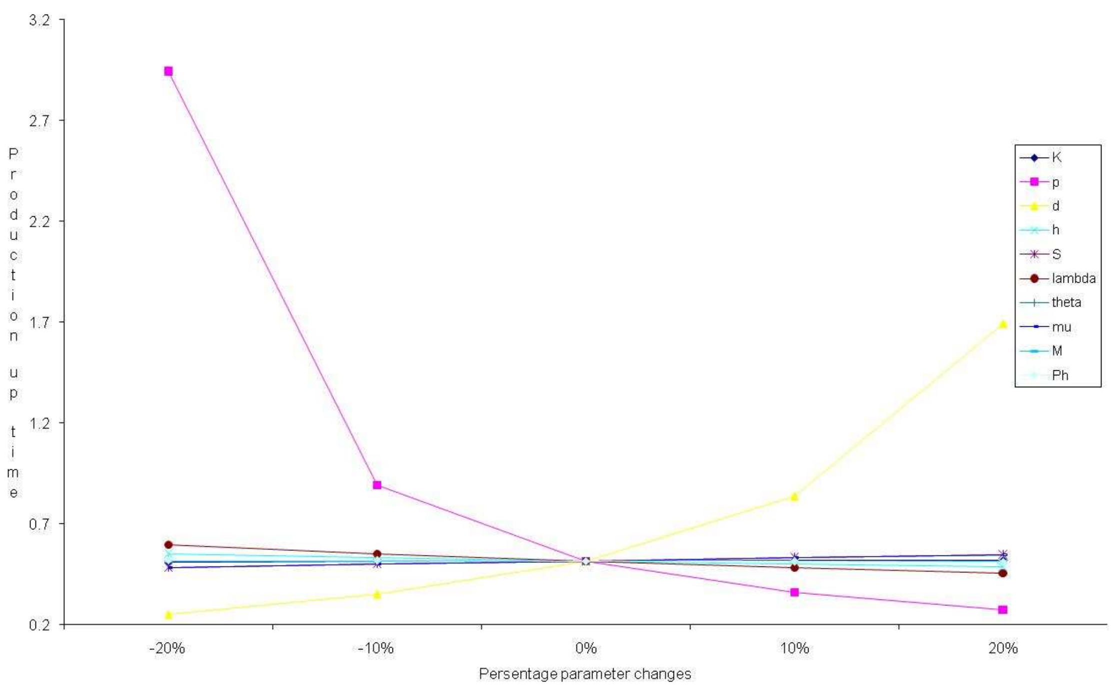

Figure 6. The T_1 sensitivity analysis for exponential distribution model

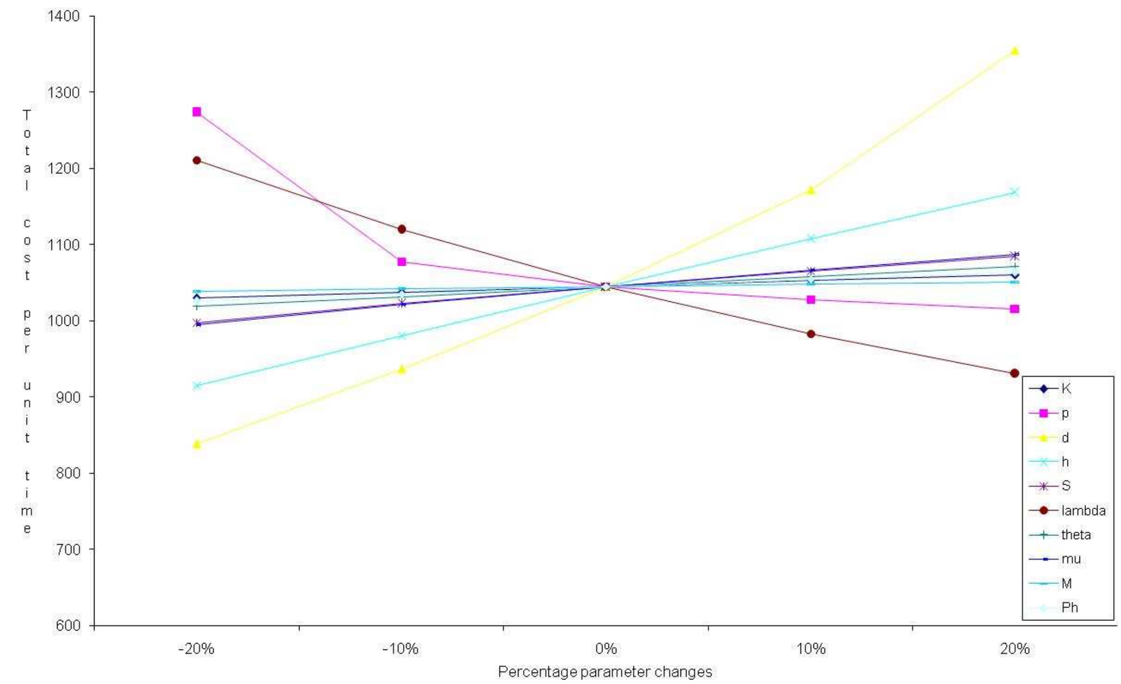

Figure 7. The total cost per unit time sensitivity analysis for exponential distribution model

4. Conclusion

The deteriorating items production inventory models with machine breakdown and stochastic repair time (for uniformly and exponentially distributed repair time) have been developed. The sensitivity analysis shows that the stochastic repair time model has a longer optimal production time than the fix repair time model when lost sales cost is high and shorter time when lost sales cost is low. The total cost difference between the stochastic model and the fix model is wider when lost sale cost increases. The optimal production time of the stochastic model is less sensitive than the fix model when the repair time varies, and the total cost gap is not as big as the total gap when lost sales cost varies. Sensitivity analysis shows that the production rate and demand rate are the most sensitive parameters for the stochastic repair time model with exponentially distributed repair time. The most sensitive parameter for the total cost is demand rate. Future research can be done to consider preventive maintenance.

References:

[1] H. Groenevelt, L. Pintelon, A. Seidman, Production lot sizing with machine breakdown, Management Science 38 (1992) 104 – 123.

[2] H. Groenevelt, L. Pintelon, A. Seidman, Production batching with machine breakdowns and safety stocks, Operations Research 40 (1992) 959 – 971.

[3] K. Moinzadeh, P. Aggarwal, Analysis of a production / inventory system subject to random disruptions, Management Science, 43 (1997) 1577 - 1588.

[4] A. Arreola-Risa, G.A. DeCroix, Inventory management under random supply disruptions and partial backorders, Naval Research Logistics 45 (1998) 687 – 703, 1998.

[5] N.E. Abboud, A simple approximation of the EMQ model with Poisson machine failures, Production Planning & Control 8 (1997) 385 – 397.

[6] B.C. Giri, W.Y. Yun, T. Dohi, Optimal design of unreliable production-inventory systems with variable production rate, European Journal of Operational Research, 162, (2005) 372 – 386.

[7] C.H. Kim, Y. Hong, An optimal production length in deteriorating production processes, International Journal of Production Economics, 58 (1999) 183 – 189.

[8] C.H. Wang, The impact of free-repair warranty policy on EMQ model for imperfect production systems, Computers & Operations Research 31 (2004) 2021 – 2035.

[9] T. Chakraborty, B.C. Giri, K.S. Chaudhuri, Production lot sizing with process deterioration and machine breakdown, European Journal of Operational Research 185 (2008) 606 – 618.

[10] L.L. Cheung, W.H. Hausman, Joint determination of preventive maintenance and safety stocks in unreliable production environment, Naval Research Logistics 44 (1997) 257 – 272.

[11] T. Dohi, H. Okamura, S. Osaki, Optimal control of preventive maintenance schedule and safety stocks in an unreliable manufacturing environment, International Journal of Production Economics 74 (2001) 147 – 155.

[12] B.C. Giri, T. Dohi, Computational aspects of an extended EMQ model with variable production rate, Computers and Operations Research 32 (2005) 3143 – 3161.

[13] S. El-Ferik, Economic production lot-sizing for an unreliable machine under imperfect age-based maintenance policy, European Journal of Operational Research 186 (2008) 150 – 163.

[14] G.L. Liao, Y.C. Chen, S.H. Sheu, Optimal economic production quantity policy for imperfect process with imperfect repair and maintenance, European Journal of Operational Research195 (2009) 348 – 357.

[15] S.W. Chiu, S.L. Wang, Y.S.P. Chiu, Determining the optimal run time for EPQ model with scrap, rework, and stochastic breakdowns, European Journal of Operational Research 180 (2007) 664 - 676.

[16] Z.T. Balkhi, The effects of learning on the optimal production lot size for deteriorating and partially backordered items with time varying demand and deterioration rates, Applied Mathematical Modelling 27 (2003) 763 – 779.

[17] C.T. Chang, L.Y. Ouyang, J.T. Teng, An EOQ model for deteriorating items under supplier credits linked to ordering quantity, Applied Mathematical Modelling 27 (2003) 983 – 996.

[18] A. Roy, M.K. Maiti, S. Kar, M. Maiti, An inventory model for a deteriorating item with displayed stock dependent demand under fuzzy inflation and time discounting over a random planning horizon, Applied Mathematical Modelling 33 (2009) 744 – 759.

[19] R.B. Misra, Optimal production lot-size inventory model for deteriorating inventory, International Journal of Production Research 15 (1975) 495 – 505.

[20] H.M. Wee, Optimal production policy for perishable items with partial backorder, Engineering Optimization 23 (1998) 315 – 322.

[21] J.J. Liao, On an EPQ model for deteriorating items under permissible delay in payments, Applied Mathematical Modelling 31 (2007) 393 – 403.

[22] H.K. Alfares, S.N. Khursheed, S.Y. Noman, Integrating quality and maintenance decisions in a production-inventory model for deteriorating items, International Journal of Production Research 43 (2005) 899 – 911.

[23] S.Y. Lo, H.M. Wee, W. C. Huang, An integrated production-inventory model with imperfect production processes and Weibull distribution deterioration under inflation, International Journal of Production Economics 106 (2007) 248 – 260.

[24] G.C. Lin, D.C Gong, On a production-inventory system of deteriorating items subject to random machine breakdowns with a fixed repair time, Mathematical and Computer modeling 43 (2006) 920 – 932.

[25] P.C. Yang, H.M. Wee, A collaborative inventory system with permissible delay in payment for deteriorating items, Mathematical and Computer Modelling 43 (2006) 209 – 221.

Appendix A

Using property 1, we set *f(x)* = the total cost per unit time, *g(x)* = the total cost and *h(x)* = the total replenishment time. The total cost per unit time is convex if the total cost is convex and the total replenishment time is concave.

The total cost of uniform distribution repair time is:

$$TC_U = K + M(1-e^{-\mu T_1}) + h\frac{p(p-d)\left(1-e^{-\mu T_1}-\mu T_1 e^{-\mu T_1}\right)}{\mu^2 d} + \pi\theta\frac{(p-d)\theta\left(1-e^{-\mu T_1}-\mu T_1 e^{-\mu T_1}\right)}{\mu^2}$$
$$+ Sd\left(\frac{\left((b\mu d)^2 - 2b(p-d)\mu d + (p-d)^2\right)\left(1-e^{-\mu T_1}\right) - e^{-\mu T_1}\mu T_1(p-d)\left((p-d)(2+\mu T_1) - 2b\mu d\right)}{2b\mu^2 d^2}\right) \qquad \text{(A1)}$$

The second derivative of (A1) with respect to T_1 is:

$$\frac{\partial TC_U^2}{\partial^2 T_1} = -M\mu^2 e^{-\mu T_1} + \left(\frac{hp}{d} + \pi\theta\right)\left((p-d)(1-b\mu)e^{-\mu T_1}\right)$$
$$-S\left(\frac{(e^{-\mu T_1})\mu^4(bd(bd + 2T_1(d-p) + T_1^2(p-d)^2) + 2(e^{-\mu T_1})\mu^3(bd(p-d) - 2T_1((p-d)^2))}{2db\mu^2}\right) \qquad \text{(A2)}$$

Set $T_1 = 0$, one has:

$$\frac{\partial TC_U^2}{\partial^2 T_1} = -M\mu^2 + (p-d)\left(\frac{hp}{d} + \pi\theta\right) - \frac{S\mu bd(\mu bd + 2(p-d))}{2} \qquad \text{(A3)}$$

Equation (A3) is convex if:

$$M < \frac{(p-d)\left(\frac{hp}{d} + \pi\theta\right) - \frac{S\mu bd(\mu bd + 2(p-d))}{2}}{\mu^2} \qquad \text{(A4)}$$

Set $T_1 = \frac{db\left(1-\frac{\theta b}{2}\right)}{(p-d)}$, one has:

$$\frac{\partial TC_U^2}{\partial^2 T_1} = \frac{1}{d}\left(h\left(8p\left(p-d\right) - 4pb\mu d(2+b\theta)\right) - S\left(4\mu bd\theta\left(p-d\right) + b\left(\mu bd\theta\right)^2\right) + \pi\left(4\theta d\left(2p - 2d - 2pdb\mu - \mu b^2 d\theta\right)\right) - M\mu^2 d\right)e^{\left(-\frac{\mu db(2+\theta b)}{2(p-d)}\right)} \qquad \text{(A5)}$$

Equation (A5) convex if:

$$M < \frac{\left(hp + \pi d\theta\right)\left(8(p-d) - 4b\mu d(2+b\theta)\right) - S\mu bd\theta\left(4(p-d) + \mu b^2 d\theta\right)}{\mu^2 d} \qquad \text{(A6)}$$

The total cost of uniform distribution repair time is convex in $0 \leq T_1 \leq \frac{db\left(1-\frac{\theta b}{2}\right)}{(p-d)}$ when (A4) and (A6) are fulfilled.

The expected replenishment time of uniform distribution repair can be written as follows:

$$E(T) = \frac{p(1-e^{-\mu T_1})}{d\mu} + Sd\left(\frac{\left((b\mu d)^2 - 2b(p-d)\mu d + (p-d)^2\right)\left(1-e^{-\mu T_1}\right) - e^{-\mu T_1}\mu T_1(p-d)\left((p-d)(2+\mu T_1) - 2b\mu d\right)}{2b\mu^2 d^2}\right) \qquad \text{(A7)}$$

The second derivative of (A7) with respect to T_1 is:

$$\frac{\partial TC}{\partial^2 T_1} = -\frac{p\mu e^{-\mu T_1}}{d} - S\left(\frac{(e^{-\mu T_1})\mu^4\left(bd(bd + 2T_1(d-p)) + T_1^2(p-d)^2\right) + 2(e^{-\mu T_1})\mu^3\left(bd(p-d) - 2T_1((p-d)^2)\right)}{2db\mu^2}\right) \qquad \text{(A8)}$$

Set $T_1 = 0$, one has:

$$\frac{\partial T_U^2}{\partial^2 T_1} = -\frac{\mu(2(p-d) + 2p + \mu bd)}{2d} \qquad \text{(A9)}$$

Equation (A9) is concave since $d < p$.

Set $T_1 = \frac{db\left(1-\frac{\theta b}{2}\right)}{(p-d)}$, one has:

$$\frac{\partial T_U^2}{\partial^2 T_1} = -\frac{\left(4p(2-\theta b) + \theta bd(4 + b^2\mu\theta)\right)\mu e^{\left(-\frac{\mu bd(2+\theta b)}{2(p-d)}\right)}}{8d} \qquad \text{(A10)}$$

Equation (A10) is concave if: $\theta b < 2$. So the expected total time of uniform distribution repair time is concave in $0 \leq T_1 \leq \frac{db\left(1-\frac{\theta b}{2}\right)}{(p-d)}$ if $\theta b < 2$.

Appendix B

The expected replenishment time of exponential distribution repair time can be written as follows:

$$E(T_E) = \frac{p(1-e^{-\mu T_1})}{d\mu} + \left(\frac{-e^{\lambda T_1\left(0.5\theta T_1\left(\frac{p-d}{d}\right)+1\right)} + e^{\lambda T_1\left(0.5\theta T_1\left(\frac{p-d}{d}\right)+1+\frac{\mu}{\lambda}\right)}}{e^{T_1\left(\frac{\lambda p}{d}+\mu\right)}} \right) \tag{B1}$$

The second derivative of (B1) with respect to T_1 is:

$$\begin{aligned}
\frac{\partial T_E^2}{\partial^2 T_1} = \Bigg(& -\lambda\left(\theta\frac{(p-d)}{d}\right) e^{T_1\left(0.5\theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda\right)} - \left(\theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda\right)^2 e^{T_1\left(0.5\theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda\right)} \\
& + \frac{(p-d)}{d}\lambda\theta e^{T_1\left(0.5\theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda+\mu\right)} \\
& + \left(\theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda+\mu\right)^2 e^{T_1\left(0.5\theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda+\mu\right)}\Bigg) \frac{1}{e^{T_1\left(\frac{\lambda p}{d}+\mu\right)}} \\
& - 2\Bigg(-\left(\theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda\right) e^{T_1\left(0.5\theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda\right)} \\
& + \left(\theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda+\mu\right) e^{T_1\left(0.5\theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda+\mu\right)}\Bigg) \frac{\left(\frac{\lambda p}{d}+\mu\right)}{e^{T_1\left(\frac{\lambda p}{d}+\mu\right)}} \\
& + \left(-e^{T_1\left(0.5\theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda\right)} + e^{T_1\left(0.5\theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda+\mu\right)}\right) \frac{\left(\frac{\lambda p}{d}+\mu\right)^2}{e^{T_1\left(\frac{\lambda p}{d}+\mu\right)}} < 0
\end{aligned} \tag{B2}$$

Equation (B2) can be simplified as:

$$\begin{aligned}
\frac{\partial T_E^2}{\partial^2 T_1} = -\frac{p\mu e^{-\mu T_1}}{d} & \\
& + \left(-\lambda\left(\theta\frac{(p-d)}{d}\right)Z - \left(\theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda\right)^2 Z + \frac{(p-d)}{d}\lambda\theta Z e^{\mu T_1}\right. \\
& \left. + \left(\theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda+\mu\right)^2 Z e^{\mu T_1}\right)\frac{1}{e^{T_1\left(\frac{\lambda p}{d}+\mu\right)}} 2\left(-\left(\theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda\right)Z\right. \\
& \left. + \left(\theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda+\mu\right)Z e^{\mu T_1}\right)\frac{\left(\frac{\lambda p}{d}+\mu\right)}{e^{T_1\left(\frac{\lambda p}{d}+\mu\right)}} + \left(-Z+Ze^{\mu T_1}\right)\frac{\left(\frac{\lambda p}{d}+\mu\right)^2}{e^{T_1\left(\frac{\lambda p}{d}+\mu\right)}} < 0
\end{aligned} \tag{B3}$$

where:

$$Z = e^{T_1\left(0.5\theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda\right)}$$

We can rewrite (B3) as:

$$\frac{\partial T_E^2}{\partial^2 T_1} = dT_{E1} + dT_{E2}$$

where:

$$dT_{E1} = -\frac{p\mu e^{-\mu T_1}}{d} \tag{B4}$$

and

$$dT_{E2} = \left(\begin{array}{l} \lambda\left(\theta\frac{(p-d)}{d}\right)(-1+e^{\mu T_1}) - (\theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda)^2 Z + (\theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda+\mu)^2 Z e^{\mu T_1} \\ \quad +2\left(\left(\theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda\right)Z - (\theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda+\mu)Ze^{\mu T_1}\right)\left(\frac{\lambda p}{d}+\mu\right) \\ \quad + (-Z+Ze^{\mu T_1})\left(\frac{\lambda p}{d}+\mu\right)^2 \end{array} \right) \tag{B5}$$

Equation (B5) can be rewritten as:

$$dT_{E2} = \lambda\left(\theta\frac{(p-d)}{d}\right)(-1+e^{\mu T_1}) - (Z_a - C)^2 Z + (Z_b - C)^2 Z e^{\mu T_1} \tag{B6}$$

where:

$$Z_a = \theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda$$

$$Z_b = \theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda+\mu$$

$$C = \left(\frac{\lambda p}{d}+\mu\right)$$

Since $Z_b > Z_a$ then (B6) is positive, so the expected total time of exponential distribution repair time is concave if:

$$\frac{p\mu e^{-\mu T_1}}{d} > \lambda\left(\theta\frac{(p-d)}{d}\right)(-1+e^{\mu T_1}) - (Z_a - C)^2 Z + (Z_b - C)^2 Z e^{\mu T_1} \tag{B7}$$

When θT_1 is small (B7) can be rewritten as:

$$\frac{p\mu e^{-\mu T_1}}{d} > \lambda\left(\theta\frac{(p-d)}{d}\right)(-1+e^{\mu T_1}) - (\lambda - C)^2 e^{\lambda T_1} + (\lambda+\mu-C)^2 e^{(\lambda+\mu)T_1} \qquad \text{(B8)}$$

When λ, μ, and T_1 are small, and through some simplifications one has:

$$2\left(\frac{p(\lambda-0.5)-\lambda d}{d}\right) - \mu < 0 \qquad \text{(B9)}$$

Equation (B9) true if λ<0.5 or $\lambda < \frac{d\mu+p}{2(p-d)}$. So the expected time for exponential distribution time is also concave when λ, μ, and T_1 are small and λ<0.5 or $\lambda < \frac{d\mu+p}{2(p-d)}$.

The total cost model of exponential distribution repair time can be modeled as:

$$TC_E = K + M(1-e^{-\mu T_1}) + h\frac{p(p-d)\left(1-e^{-\mu T_1}-\mu T_1 e^{-\mu T_1}\right)}{\mu^2 d} + \pi\theta\frac{(p-d)\theta\left(1-e^{-\mu T_1}-\mu T_1 e^{-\mu T_1}\right)}{\mu^2}$$
$$+ Sd\left(\frac{-e^{\lambda T_1\left(0.5\theta T_1\left(\frac{p-d}{d}\right)+1\right)} + e^{\lambda T_1\left(0.5\theta T_1\left(\frac{p-d}{d}\right)+1+\frac{\mu}{\lambda}\right)}}{e^{T_1\left(\frac{\lambda p}{d}+\mu\right)}}\right) \qquad \text{(B10)}$$

The second derivative of (B10) with respect to T_1 is:

$$\frac{\partial TC_E^2}{\partial^2 T_1} = -M\mu^2 e^{-\mu T_1} + \left(\frac{hp}{d}+\pi\theta\right)(p-d)e^{-\mu T_1}(1-\mu T_1)$$
$$+ Sd\left(\left(-\lambda\left(\theta\frac{(p-d)}{d}\right)e^{T_1\left(0.5\theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda\right)} - \left(\theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda\right)^2 e^{T_1\left(0.5\theta T_1\lambda\left(\frac{p-s}{d}\right.\right.}\right.\right.$$
$$+ \frac{(p-d)}{d}\lambda\theta e^{T_1\left(0.5\theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda+\mu\right)}$$
$$\left. + \left(\theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda+\mu\right)^2 e^{T_1\left(0.5\theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda+\mu\right)}\right)\frac{1}{e^{T_1\left(\frac{\lambda p}{d}+\mu\right)}}$$
$$- 2\left(-\left(\theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda\right)e^{T_1\left(0.5\theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda\right)} + \left(\theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda\right.\right.$$
$$\left.\left. + \mu\right)e^{T_1\left(0.5\theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda+\mu\right)}\right)\frac{\left(\frac{\lambda p}{d}+\mu\right)}{e^{T_1\left(\frac{\lambda p}{d}+\mu\right)}}$$
$$\left. + \left(-e^{T_1\left(0.5\theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda\right)} + e^{T_1\left(0.5\theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda+\mu\right)}\right)\frac{\left(\frac{\lambda p}{d}+\mu\right)^2}{e^{T_1\left(\frac{\lambda p}{d}+\mu\right)}}\right)$$

(B11

)

When λ,μ, and T_1 are small, (B11) can be rewritten as:

$$\frac{\partial TC_E^2}{\partial^2 T_1} = -M\mu^2 + \left(\frac{hp}{d} + \pi\theta\right)(p-d) - Sd\left(\left(\lambda - \left(\frac{\lambda p}{d} + \mu\right)\right)^2 + \left(\lambda + \mu - \left(\frac{\lambda p}{d} + \mu\right)\right)^2\right) > 0$$

The expected total cost of exponential distribution repair time is convex if:

$$M < \frac{\left(\frac{hp}{d} + \pi\theta\right)(p-d) - Sd\left(2\left(\frac{\lambda\mu(p-d)}{d}\right) - \mu^2\right)}{\mu^2} \qquad \text{(B12)}$$